Construction of the log-convex minorant of a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$

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For a sequence $\{M_p\}_{p\in\mathbb{N}_0}$ of real positive numbers, its *associated function* is defined by

$$\omega_M(t) := M_0 \sup_{p\in\mathbb{N}_0} \log \frac{t^p}{M_p}, \qquad t>0.$$

Mandelbrojt proved in [2] that if $\lim_{p\to+\infty} M_p^{1/p} = +\infty$, then

$$M_p = M_0 \sup_{t>0} \frac{t^p}{\exp \omega_M(t)}, \qquad p\in\mathbb{N}_0, \tag{1}$$

if and only if $\{M_p\}_{p\in\mathbb{N}_0}$ is logarithmically convex, i.e.

$$M_p^2 \leq M_{p-1}M_{p+1}, \qquad \forall p\in\mathbb{N}.$$

However, condition (1) had never been generalized to the d-dimensional anisotropic case, since the classical coordinate-wise logarithmic convexity condition

$$M_\alpha^2 \leq M_{\alpha-e_j}M_{\alpha+e_j}, \qquad \alpha\in\mathbb{N}_0^d,\ 1\leq j\leq d,\ \alpha_j\geq 1,$$

is not sufficient. Assuming the stronger condition that $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is log-convex on the globality of its variables, in the sense that $\log M_\alpha = F(\alpha)$ for a convex function $F:[0,+\infty)^d\to\mathbb{R}$, we extend (1) to

$$M_\alpha = M_0 \sup_{s\in(0,+\infty)^d} \frac{s^\alpha}{\exp \omega_{\mathbf{M}}(s)}, \qquad \forall\alpha\in\mathbb{N}_0^d. \tag{2}$$

To obtain this result we construct the (optimal) convex minorant of a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ (then $a_\alpha = \log M_\alpha$) by taking the supremum of hyperplanes approaching from below the given sequence. This leads to the notion of convexity for a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ in the sense that $a_\alpha = F(\alpha)$ for a convex function F, which corresponds to the suitable notion of logarithmic convexity for a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ as stated above.

This result is a very useful tool for working in the anisotropic setting, and we expect several applications, that could be object of future works.

References

[1] C. Boiti, D. Jornet, A. Oliaro, G. Schindl, Construction of the log-convex minorant of a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$, *Math. Nachr.*, **298** (2025), 456–477.

[2] S. Mandelbrojt, Séries adhérentes, Régularisation des suites, Applications, Gauthier-Villars, Paris, 1952.