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What the shape of a domain knows about spaces of holomorphic functions

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Let Ω be a domain in the complex plane $\mathbb{C}$ and X a space of holomorphic functions in the unit disc $\mathbb{D}$. This talk aims to survey the state of the art regarding the following question: Can we characterize in terms of the geometry of Ω when every holomorphic function $f : \mathbb{D} \to \Omega$ belongs to X? That is, when $\mathrm{Hol}(\mathbb{D}, \Omega) \subset X$? In such a case, we say that Ω is a X-domain.

A classical result due to Seidel and Walsh (1942) [7] addresses this question for the Bloch space $\mathcal{B}$. Namely, they proved that a domain Ω is a $\mathcal{B}$-domain if and only if there exists a uniform upper bound for the radii of all the disks contained in Ω.

Subsequently, Hayman and Pommerenke (1978) [4] provided a characterization for Ω being a BMOA-domain, where BMOA denotes the space of analytic functions of bounded mean oscillation. It is also worth mentioning that Ahern and Cohn (1983) studied the $\mathcal{N}^+$-domains, where $\mathcal{N}^+$ is the Smirnov class.

The first to address this problem for Hardy spaces H^p was Hansen in 1968. He introduced the concept of Hardy number of a domain Ω as the supremum of all $p > 0$ for which Ω is a H^p-domain. Significant geometric characterizations related to this number were later provided by Essen (1981) [3] and Kim and Sugawa (2011) [6]. Characterizing the Hardy number for specific classes of domains remains an active area of research. In this talk, we will present recent results about the Hardy number of Koenigs domains (those satisfying that $\Omega + 1 \subset \Omega$) based on joint work with Cruz-Zamorano, Kourou, and Rodríguez-Piazza [2].

Finally, we will address recent developments concerning Bergman spaces. While the first results in this setting were initiated in a series of works of Betsakos, Karafyllia, and Karamanlis, we will focus specifically on recent results by Karafyllia [5] and Betsakos and Cruz-Zamorano [1].

A recurring theme throughout the talk will be the crucial role played by potential theory techniques in characterizing X-domains for the various spaces discussed.

References

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