*International Workshop on Functional Analysis*
*On the Occasion of the 70th Birthday of José Bonet*
*17-19 June 2025, Valencia, Spain*

# Can we detect quasicrystals using time-frequency distributions?

Paolo Boggiatto, University of Torino, Italy

joint work with
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and Alessandro Oliaro, Univ. of Torino

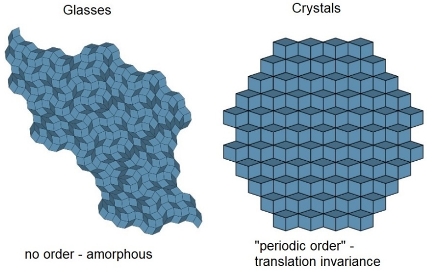
# PLAN OF THE TALK

# AN INTRIGUING HISTORY...

WHAT IS A QUASICRYSTAL?

In mineralogy:

- Till the 80's: solid state matter in 2 forms, unordered and ordered (crystals)



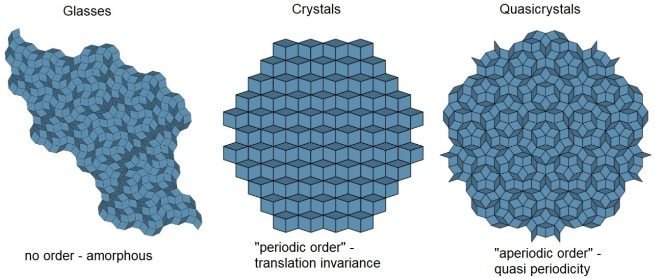

https://matmatch.com/resources/blog/quasicrystals-materials-that-should-not-exist/

# AN INTRIGUING HISTORY...

WHAT IS A QUASICRYSTAL?

In mineralogy:

- Till the 80's: solid state matter in 2 forms, unordered and ordered (crystals)
- 1982: Daniel Schechtman discovers an alluminium-manganese alloy with prohibited diffraction patterns. Though initially contrasted, wins the Nobel Prize in 2011.

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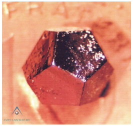

Koryak Mountains - Kamchatka Peninsula
Bindi-Steinhardt expedition 2011

Many have contributed to a mathematical theory of quasicrystals:

- R. Penrose, Y. Meyer (1970) (preceding Schechtman discovery!),
- A. Cordoba (1989),
- Kolountzakis (1996)
- Lagarias (1996),
- S. Yu. Favorov (2016)
- V.P. Palamodov (2017)
- P. Kurasov and P. Sarnak (2020)
- ...and others

# FOURIER QUASICRYSTALS

Tilings by "Cut and Project" method: **Fibonacci Quasicrystal**, a 1-dimensional model

S = small rabbit    L = large rabbit

| n. | rabbits |
|---|---|
| 1 | S |
| 1 | L |
| 3 | LS |
| 3 | LSL |
| 5 | LSLLS |
| 8 | LSLLSLSL |
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| | etc. |

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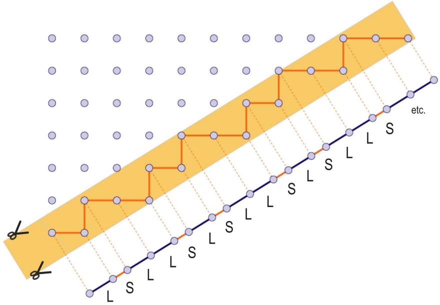


Strip of irrational slope $\frac{1}{\tau}$ and width $\frac{1+\tau}{\sqrt{2+\tau}}$

($\tau = \frac{1+\sqrt{5}}{2}$ "golden section")

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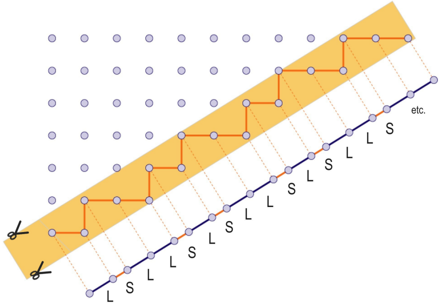


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This method, which can be abstractely formalized, leads to distributions of the type

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$$

where $\Lambda$ is in some sense a "quasi-periodic" discrete set, and with Fourier transform of the same type.

# FOURIER QUASICRYSTALS

There is not a univocal definition of *quasicrystals*, we focus on that of *Fourier quasicrystals* ...even for the definition of Fourier quasicrystals there is not a perfect agreement, however usually the following is assumed (Lev, Olevskii):

## Fourier quasicrystals

A temprered distribution $\mu \in \mathcal{S}'(\mathbb{R}^d)$ is a called *Fourier quasicrystal* if $\mu$ and $\widehat{\mu}$ are of the form

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda, \qquad \widehat{\mu} = \sum_{s \in S} b_s \delta_s,$$

where $\Lambda$ and $S$ are discrete subsets of $\mathbb{R}^d$, and $\delta_\xi$ is the mass point at $\xi$.
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## Basic examples: Dirac combs

Let $L = A(\mathbb{Z}^d)$ be a full-rank lattice (i.e. the matrix $A$ is non-degenerate), and let $L^* = \{\lambda^* \in \mathbb{R}^d : \langle \lambda^*, \lambda \rangle \in \mathbb{Z}\}$ the *dual* lattice. Then for the *Dirac comb* $\delta_L = \sum_{\lambda \in L} \delta_\lambda$ we have $\widehat{\delta_L} = \frac{1}{\det A} \delta_{L^*}$.

Clearly Dirac combs have a well-defined periodic structure.
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**Question (Lagarias 2000):** is "part of this structure in some sense" also present in Fourier quasicrystals?
**Answer (Lev, Olevskii 2015 and 2016):** one "positive", and one "negative" result.

### Definition

A set $A \subset \mathbb{R}^d$ is said to be *uniformly discrete* (u.d.) if there is $\delta > 0$ such that $|r - s| \geq \delta$ whenever $s, r \in A, s \neq r$.

### Theorem ("positive") (N. Lev, A. Olevskii, (Inventiones Math. 2015))

If a measure $\mu$ on $\mathbb{R}^d$ (which is assumed to be positive in the case $d > 1$) is a Fourier quasicrystal and both the support and the spectrum of $\mu$ are u.d. then there are a lattice $L$ on $\mathbb{R}^d$, vectors $\theta_j \in \mathbb{R}^d$ and trigonometric polynomials $P_j$ $(1 \leq j \leq N)$ such that

$$\mu = \sum_{j=1}^{N} \sum_{\lambda \in L + \theta_j} P_j(\lambda)\delta_\lambda. \tag{1}$$

The same holds for $\widehat{\mu}$ (with the dual lattice).

## Remark

REMARK: the previous theorem shows that $\mu$ is a sum of translations and modulations of the Dirac delta. Viceversa any tempered distribution of the type (1) has both u.d. support and spectrum.

Note that the "quasi-periodicity" of $\mu$ is only due to the polynomials $P_j(x) = \sum_{k=-N_j}^{N_j} a_k^{(j)} e^{2\pi i \omega_k^{(j)} x}$.

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Another "positive" result:

## Theorem (V.P. Palamodov, (JFAA 2017))

Suppose that $0 \neq \mu \in \mathcal{S}'(\mathbb{R}^d)$ has support $\Lambda$ and spectrum $\Sigma$ such that $\Lambda - \Lambda$ and $\Sigma - \Sigma$ are discrete and (at least) one of them is u.d. Then for $\mu$ and $\widehat{\mu}$ the same conclusion holds as in the previous theorem

The uniform discreteness of the support and the spectrum of $\mu$ in the previous theorem is essential as proved by the following:

**Theorem ("negative") (N. Lev, A. Olevskii (Rev. Mat. Iberoam. 2016))**

There exists a Fourier quasicrystal $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ with $\widehat{\mu} = \sum_{s\in S} b_s \delta_s$ such that $\Lambda$ and $S$ are discrete closed sets, but $\Lambda$ contains only finitely many elements of any arithmetic progression. The construction can easily be extended to $n > 1$.

It follows that $\mathrm{supp}\,\mu$ can not contain any lattice and in this sense $\mu$ is someow "far away" from being periodic.

# WIGNER TRANSFORMS AND QUASICRYSTALS

In view of the symmetric conditions on $\mu$ and $\widehat{\mu}$ in the definition of Fourier quasicrystals, it seems reasonable that some information about the quasi-periodic structure of $\mu$ can be deduced from the knowledge of its time-frequency distribution.
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We consider the Wigner transform

The cross-Wigner distribution of function or distributions $f, g$ on $\mathbb{R}^d$ is

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega t}dt, \quad x,\omega \in \mathbb{R}^d.$$

It defines sesquilinear maps in the functional settings:

$$W : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^{2d})$$
$$W : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^{2d})$$
$$W : \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^{2d}) \quad \text{(...and many others)}$$

The Wigner transform of $f$ is $W(f) := W(f,f)$

$W(f)$ is a quadratic representation of the signal $f$ giving information about the energy of the signal $f$ with respect to both time and frequency.

## An negative but interesting start...

The most natural condition is to consider the case when $W(\mu)$ is supported on a u. d. set of $\mathbb{R}^{2d}$. However, due to the interaction between the Wigner distribution and the metaplectic operators, from this fact we cannot even deduce that the support of $\mu$ is discrete.

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Actually, to every symplectic matrix $A \in \mathrm{Sp}(2, \mathbb{R})$ let $T_A$ be a unitary operator acting on $L^2(\mathbb{R})$ such that

$$W\big(T_A f, T_A g\big)(z) = W(f, g)(A^* z) \quad \forall z = (x, \omega) \in \mathbb{R}^2. \tag{2}$$

$T_A$ extends to an isomorphism on $\mathcal{S}'(\mathbb{R})$, and (2) holds for $f, g \in \mathcal{S}'(\mathbb{R})$.

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**Example (P.B., C. Fernandez, A. Galbis, A. Oliaro, (JFA 2022))**

Let us consider $f = g = \mu = \sum_{n \in \mathbb{Z}} \delta_n$ (the usual Dirac comb):
When

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \; \theta \in (-\pi, \pi)$$

$T_A$ is the fractional Fourier transform $\mathcal{F}^\alpha$, where $\alpha = \frac{2}{\pi}\theta \in (-2, 2)$. In this case $W(\mu)(z)$ is a well-known distribution with u.d. support and $W(T_A\,\mu)(z) = W(\mu)(A^* z)$ is a rotation of $W(\mu)$ hence also supported on a u.d. set of $\mathbb{R}^2$. However for a suitable choice of $\theta$ we have $\operatorname{supp}\, T_A\,\mu = \mathbb{R}$

We have therefore to assume more restrictive conditions on the support of $W\mu$:
our assumption will be that $\mu \in \mathcal{S}'(\mathbb{R}^d)$ has Wigner transform $W(\mu)$ supported on the product of two uniformly discrete subsets of $\mathbb{R}^d$.

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Our result is the following

**Theorem (P.B., C. Fernandez, A. Galbis, A. Oliaro, (JFA 2022)) (*)**

Let $\mu \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $W(\mu) = \sum_{(r,s)\in A\times B} c_{r,s}\delta_{(r,s)}$ where $A, B$ are uniformly discrete sets in $\mathbb{R}^d$. Then $\mu$ and $\widehat{\mu}$ are measures with supports contained in $A$ and $B$ respectively, i.e. $\mu$ is a quasicrystal with u.d. support and spectrum.

By the Lev-Olevskii and Palamodov "positive" results we have then

$$\mu = \sum_{j=1}^{N} \sum_{\lambda \in L+\theta_j} P_j(\lambda)\delta_\lambda,$$

the same holds for $\widehat{\mu}$.

### Lemma

*Let $\mu \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $W(\mu) = \sum_{(r,s)\in A\times B} c_{r,s}\delta_{(r,s)}$ where $A, B$ are u.d. sets. Then supp $\mu \subset A$ and supp $\widehat{\mu} \subset B$.*

*Moreover, $\frac{r_1+r_2}{2} \in A$ for any $r_1, r_2 \in$ supp $\mu$, and similarly $\frac{s_1+s_2}{2} \in B$ for any $s_1, s_2 \in$ supp $\widehat{\mu}$*

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## Remark

Note that the inclusions obtained above go into the "opposite" direction with respect to classical inclusions

$$\Pi_1(\text{supp}\, W\mu) \subseteq H(\text{supp}\, \mu), \qquad \Pi_2(\text{supp}\, W\mu) \subseteq H(\text{supp}\, \widehat{\mu})$$

where $\Pi_j$ are the projections from $\mathbb{R}^d \times \mathbb{R}^d$ onto the first and second factor, and $H$ indicates the convex hall of a set.

# Idea of the proof of Theorem (*) in dimension $d = 1$:

• As $\mu \in \mathcal{S}'(\mathbb{R}^d)$ it is of finite order, and from the previous Lemma $\operatorname{supp}\mu \subseteq A$ which is u.d., therefore, by the structure theorem, there exists $N \in \mathbb{N}$ such that

$$\mu = \sum_{r \in \operatorname{supp}\mu} \sum_{j=0}^{N} a_r^j \delta_r^{(j)},$$

with $a_r^j \in \mathbb{C}$. We assume $N \geq 1$ and want to show that $a_r^N = 0$ for all $r \in \operatorname{supp}\mu$.

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• For $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R})$ we explicitly compute $\langle W(\mu), \phi_1 \otimes \phi_2 \rangle$

• We use now the uniform discreteness: For each $r_0 \in \operatorname{supp}\mu$ we set $\phi_1(x) = \psi(t(x - r_0))$, with $t > 1$, and for suitably localized $\psi$ we get

$$t^{2N} \left| \sum_{(r,s) \in D(r_0)} \overline{a_r^N} a_s^N \overline{\widehat{\phi}_2}(r - s) \right| \leq C ||\psi||_\infty ||\phi_2||_\infty,$$

with $D(r_0) = \{(r, s) \in \operatorname{supp}\mu : \frac{r+s}{2} = r_0\}$

- For $t \longrightarrow \infty$ we get $\sum_{(r,s)\in D(r_0)} \overline{a_r^N} a_s^N \overline{\widehat{\phi_2}}(r-s) = 0$ for every $\phi_2 \in \mathcal{D}(\mathbb{R})$.

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- The conclusion for $\widehat{\mu}$ now follows from $W(\widehat{\mu})(x, \omega) = W(\mu)(-\omega, x)$. $\square$

# MATRIX-WIGNER TRANSFORMS

The Wigner transform can be decomposed into three steps:

$$f, g \longrightarrow f \otimes \overline{g} \longrightarrow \tau(f \otimes \overline{g}) \longrightarrow W(f,g) = \mathcal{F}_2(\tau(f \otimes \overline{g}))$$

where

$$\begin{aligned}
(f \otimes \overline{g})(x,t) &= f(x)\overline{g(t)}, \\
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Let's replace the "torsion" $\tau$ in the second step by a general change of variable:

**Definition**

The *Matrix-Wigner transform* of $f, g \in \mathcal{S}'(\mathbb{R}^d)$ associated with $T \in GL(2d, \mathbb{R})$ is

$$W_T(f,g) = \mathcal{F}_2(T(f \otimes \overline{g})).$$

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- It includes most of the basic time-frequency representations
- Unifying framework where we can focus the connections between our results and those of Lev-Olevskii.

Let us take $T : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ invertible linear transformation with inverse

$$T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

**Proposition (P.B., C. Fernandez, A. Galbis, A. Oliaro, (JFA 2022))**

Let $\Pi_1 \mathrm{supp}\, W_T(\mu,\nu)$ be a u.d. set, then $\begin{cases} \det A \neq 0 \Longrightarrow \mathrm{supp}\,\mu \text{ is u.d,} \\ \det B \neq 0 \Longrightarrow \mathrm{supp}\,\nu \text{ is u.d.} \end{cases}$

Let $\Pi_2 \mathrm{supp}\, W_T(\mu,\nu)$ be a u.d. set, then $\begin{cases} \det A \neq 0 \Longrightarrow \mathrm{supp}\,\widehat{\nu} \text{ is u.d,} \\ \det B \neq 0 \Longrightarrow \mathrm{supp}\,\widehat{\mu} \text{ is u.d.} \end{cases}$

Some particular cases:

### Wigner

If $T^{-1} = \begin{pmatrix} \frac{1}{2}\mathrm{Id} & \frac{1}{2}\mathrm{Id} \\ \mathrm{Id} & -\mathrm{Id} \end{pmatrix}$ then $W_T(\mu,\nu) = W(\mu,\nu)$ Wigner transform.
$\Pi_1 \mathrm{supp}\, W(\mu,\nu)$ u.d. $\Rightarrow \mathrm{supp}\,\mu, \mathrm{supp}\,\nu$ u.d.
(resp. $\Pi_2 \mathrm{supp}\, W(\mu,\nu)$ u.d. $\Rightarrow \mathrm{supp}\,\widehat{\mu}, \mathrm{supp}\,\widehat{\nu}$ u.d)

Some particular cases:

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## Ambiguity Function

If $T^{-1} = \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ \frac{1}{2}\mathrm{Id} & \frac{1}{2}\mathrm{Id} \end{pmatrix}$ then
$W_T(\mu,\nu) = \int_{\mathbb{R}^d} e^{-2\pi i\omega t}\mu(t+x/2)\overline{\nu(t-x/2)}\,dt = A(\mu,\nu)$ is the *Ambiguity function* and the same conslusion holds as for Wigner.

(and similarly for the STFT, $\tau$-Wigner, and many other time-frequency representations)

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## Lev-Olevskii ($\simeq$ Rihaczek transform)

If $T^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$ then $W_T(\mu,\nu) = \mathcal{F}_2(\mu\otimes\overline{\nu})(x,\omega) = \mu(x)\overline{\widehat{\nu}(\omega)}$,
therefore Lev-Olevskii hypothesis $\mu = \sum_{\alpha\in\Lambda} a_\alpha\delta_\alpha$; $\quad \widehat{\mu} = \sum_{\beta\in S} b_\beta\delta_\beta$ with $\Lambda, S$ u.d. sets is a particular case of the hypothesis $W_T(\mu,\nu) = \sum_{(r,s)\in A\times B} c_{r,s}\delta_{(r,s)}$ with $A, B$ u.d. sets.

Under slightly more strict conditions on the matrix we have the following generalization of the Lev-Olevski theorem (in dimension $d = 1$):

**Proposition (P.B., C. Fernandez, A. Galbis, A. Oliaro, (JFA 2022))**

Let $\mu, \nu \in \mathcal{S}'(\mathbb{R}) \setminus \{0\}$.
Suppose there exist $T : \mathbb{R}^2 \to \mathbb{R}^2$ invertible linear transformation with inverse

$$T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a \neq 0, b \neq 0,$$

such that

$$W_T(\mu, \nu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)} \quad \text{with } A, B \text{ u.d. sets.}$$

Then $\mu, \nu, \widehat{\mu}, \widehat{\nu}$ are measures supported in u.d. sets, i.e.

$$\mu = \sum_{r \in S_\mu} a_r \delta_r, \qquad \nu = \sum_{s \in S_\nu} b_s \delta_s, \qquad \widehat{\mu} = \sum_{r \in S_{\widehat{\mu}}} \tilde{a}_r \delta_r, \qquad \widehat{\nu} = \sum_{s \in S_{\widehat{\nu}}} \tilde{b}_s \delta_s,$$

for some u.d. sets $S_\mu, S_\nu, S_{\widehat{\mu}}, S_{\widehat{\nu}}$ (and slowly increasing coefficients $a_j, b_j, \tilde{a}_j, \tilde{b}_j \in \mathbb{C}$).

It follows that $\mu$ and $\nu$ are of the form $\sum_{j=1}^{N} \sum_{\lambda \in L + \theta_j} P_j(\lambda) \delta_\lambda$ (for some lattices $L$, trigonometric polynomials $P_j$, and $\theta_j \in \mathbb{R}$).

Finally in general dimension $d$ the situation is described by the following two complementary and mutually exclusive theorems:

Let $T$ be an invertible $2d \times 2d$ matrix of the form

$$T = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}, \qquad \text{with inverse} \qquad T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and suppose that $\mu \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $W_T(\mu) = \sum_{(r,s)\in R\times S} c_{r,s}\delta_{(r,s)}$ with $R, S \subset \mathbb{R}^d$ u.d. sets.

**Theorem (1) (P.B., C.Fernandez, A.Galbis, A.Oliaro; Res. Math., 2025)**

If

- $\sup_{s\in S} |c_{rs}| < \infty$ for every $r \in R$ and $\sup_{r\in R} |c_{rs}| < \infty$ for every $s \in S$;
- $\det(B_0 - D_0) \neq 0$;
- $\det(A + B) \neq 0$.

Then $\mu$ and $\widehat{\mu}$ are measures whose supports, $\Lambda$ and $\Sigma$, are uniformly discrete. If furthermore $W_T$ is in the Cohen class, then $\Lambda \subseteq R$ and $\Sigma \subseteq S$.

Main examples: Cohen class representations and in particular the Wigner transform

Finally in general dimension $d$ the situation is described by the following two complementary and mutually exclusive theorems:

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and suppose that $\mu \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $W_T(\mu) = \sum_{(r,s)\in R\times S} c_{r,s}\delta_{(r,s)}$ with $R, S \subset \mathbb{R}^d$ **discrete** (not necessarily u.d.) sets.

**Theorem (2) (P.B., C.Fernandez, A.Galbis, A.Oliaro; Res. Math., 2025)**

- If $B_0 = D_0$ then $\mu$ and $\widehat{\mu}$ are measures whose supports, $\Lambda$ and $\Sigma$, are uniformly discrete; moreover, there exist invertible $d \times d$ matrices $M$ and $N$ such that $\Lambda - \Lambda \subset M(R)$, $\Sigma - \Sigma \subset N(S)$.
- If moreover $R$ or $S$ is uniformly discrete then $\mu = \sum_{j=1}^{N} P_j \sum_{\lambda\in L+\theta_j} \delta_\lambda$, where $L$ is a lattice, $\theta_j \in \mathbb{R}^d$ and $P_j(x)$ is a trigonometric polynomial, i.e. $\mu$ is a Fourier quasicrystal.

Main example: Ambiguity function

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THANK YOU !!!

AND HAPPY BIRTHDAYS PEPE!!!