Construction of the log-convex minorant of a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$

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Joint work:

C. Boiti, D. Jornet, A. Oliaro, G. Schindl, *Construction of the log-convex minorant of a sequence* $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$, Math. Nachr. **298** (2025), 456-477.

Motivation: ultradifferentiable setting

Given

- a sequence $\{M_p\}_{p\in\mathbb{N}_0}$ of real positive numbers $(\mathbb{N}_0 = \mathbb{N}\cup\{0\})$
- an open subset Ω in $\mathbb{R}^d$

Ultradifferentiable function of class M_p

$f \in C^\infty(\Omega)$ is said to be an ultradifferentiable function of class M_p if for every compact $K \subset\subset \Omega$

$$\sup_K |D^\alpha f| \leq Ch^{|\alpha|} M_{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^d$$

Beurling case (ultradifferentiable of class (M_p)): $\forall h > 0\ \exists C > 0$
Roumieu case (ultradifferentiable of class $\{M_p\}$): $\exists h > 0,\ C > 0$

Isotropic case

Derivatives estimated in terms of $M_{|\alpha|}$

Anisotropic case

Estimate derivatives in terms of M_α

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Associated function in the one-dimensional case

Associated function

$$\omega_M(t) := M_0 \sup_{p\in\mathbb{N}_0} \log \frac{t^p}{M_p}, \qquad t > 0.$$

If $\lim_{p\to+\infty} M_p^{1/p} = +\infty$, then[1]

Proposition (Mandelbrojt, 1952)

$$M_p = M_0 \sup_{t>0} \frac{t^p}{\exp \omega_M(t)}, \qquad p \in \mathbb{N}_0,$$

if and only if $\{M_p\}_{p\in\mathbb{N}_0}$ is logarithmically convex, i.e.

$$M_p^2 \le M_{p-1}M_{p+1}, \qquad \forall p \in \mathbb{N}. \tag{1}$$

[1] S. Mandelbrojt, *Séries adhérentes, Régularisation des suites, Applications*, Gauthier-Villars, Paris, 1952.
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Multi-dimensional case

d-dimensional case

The analogous result for a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$

$$M_\alpha = M_0 \sup_{s\in(0,+\infty)^d} \frac{s^\alpha}{\exp\omega_{\mathbf{M}}(s)}, \qquad \forall\alpha\in\mathbb{N}_0^d, \tag{2}$$

had been never proved before

Why?

The classical coordinate-wise logarithmic convexity condition

$$M_\alpha^2 \le M_{\alpha-e_j}M_{\alpha+e_j}, \qquad \alpha\in\mathbb{N}_0^d,\ 1\le j\le d,\ \alpha_j\ge 1, \tag{3}$$

for a sequence $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is not sufficient to obtain (2).

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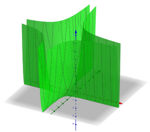
Example

$M_\alpha := e^{F(\alpha)}$ for $F(x,y)=(x+1)^2(y+1)^2$ satisfies (3) but not (2)

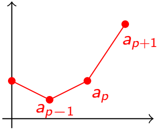


$F(x,y)$ not convex in $[0,+\infty)^2$

Convexity

Convex sequence (1-dimensional case)

$\{a_p\}_{p\in\mathbb{N}}$ is convex if $a_p \le \frac{1}{2}a_{p-1} + \frac{1}{2}a_{p+1}$ $\forall p \in \mathbb{N}$



$\Leftrightarrow$ the polygonal obtained by connecting the points (p, a_p) is the graph of a convex function
$\Leftrightarrow$ $\exists$ a convex function $F : [0, +\infty) \to \mathbb{R}$ with $F(p) = a_p$ $\forall p \in \mathbb{N}_0$

N.B. $a_p = \log M_p$ is convex iff $\{M_p\}$ logarithmically convex: $M_p^2 \le M_{p-1}M_{p+1}$

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$\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is logarithmically convex if $a_\alpha = \log M_\alpha$ is convex:
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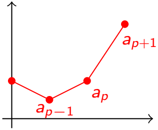

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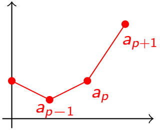

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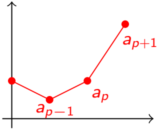

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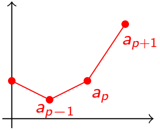

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Ex: $M_\alpha := e^{F(\alpha)}$
$F(x, y) = (x + 1)^2(y + 1)^2$
is not logarithmically convex

(Logarithmically) convex minorant

Given $\{M_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ with $\lim_{|\alpha|\to+\infty} M_\alpha^{1/|\alpha|}=+\infty$ (and $M_0=1$),

the idea is now to construct the largest logarithmically convex sequence $\{M_\alpha^{\mathrm{lc}}\}_{\alpha\in\mathbb{N}_0^d}$ with $M_\alpha^{\mathrm{lc}}\leq M_\alpha$ and prove that

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We shall do the construction for $a_\alpha=\log M_\alpha$

The convex minorant of a sequence $\{a_\alpha\}_{\alpha\in\mathbb{N}_0^d}$ is the largest convex sequence $\{a_\alpha^c\}_{\alpha\in\mathbb{N}_0^d}$ with $a_\alpha^c\leq a_\alpha$ for all $\alpha\in\mathbb{N}_0^d$

Assumptions for the construction

- $a_\alpha>-\infty,\quad \forall\alpha\in\mathbb{N}_0^d$
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Construction by hyperplanes

$S := \{(\alpha, a_\alpha) : \alpha \in \mathbb{N}_0^d\}$

$\mathcal{L} := \{f : \mathbb{R}^d \to \mathbb{R} : f \text{ is an affine function}\} = \{f(x) = \langle k, x\rangle + c : k \in \mathbb{R}^d, c \in \mathbb{R}\}$

$\mathcal{L}_S := \{f \in \mathcal{L} : f(\alpha) \le a_\alpha \ \forall \alpha \in \mathbb{N}_0^d\}$ hyperplanes that lie under S

IDEA:

Take the supremum of the hyperplanes that lie under S

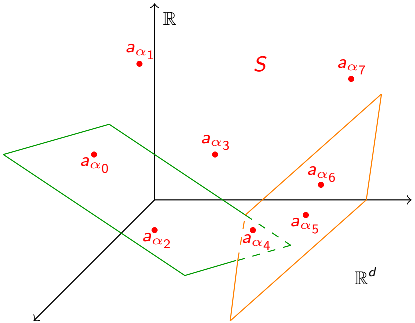
$$F(x) := \sup_{f \in \mathcal{L}_S} f(x)$$

and project each (α, a_α) to get the convex minorant sequence

$$a_\alpha^c = F(\alpha) \quad (\le a_\alpha)$$

N.B. F in convex

How to prove that it is the largest convex function $\le a_\alpha$?



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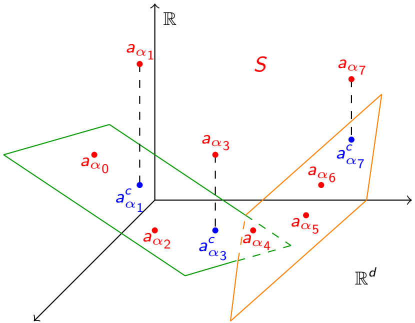

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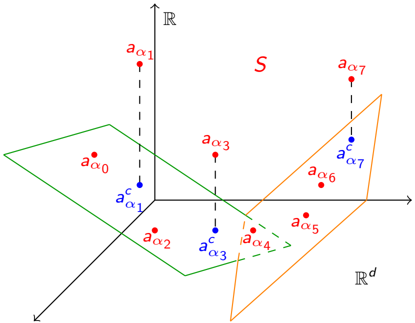

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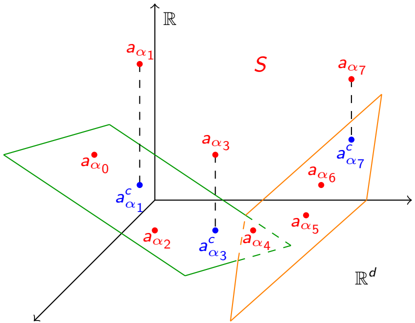

Geometric construction in the 1-dimensional case

Hyperplanes that lie under S: straight lines

$f \in \mathcal{L}_S \Leftrightarrow f(x) = kx + c$ with $k, c \in \mathbb{R}$ and $f(0) = c \le a_0$

Take $f_{a_0,k}(x) = a_0 + kx$ ($a_0 \in \mathbb{R}$)
Rotate around $(0, a_0)$ until we meet another point $(p, a_p) \in S$: $a_p = a_0 + kp \Rightarrow$
choose $k_0 := \inf_{p\in\mathbb{N}} \frac{a_p - a_0}{p} = \min_{p\in\mathbb{N}} \frac{a_p - a_0}{p} = \frac{a_{p_1} - a_0}{p_1}$ (since $\frac{a_p}{p} \to +\infty$)

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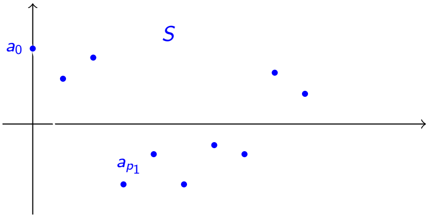

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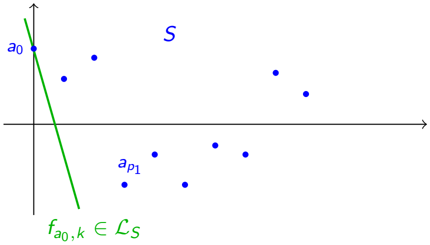

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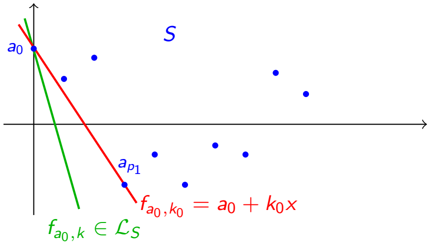

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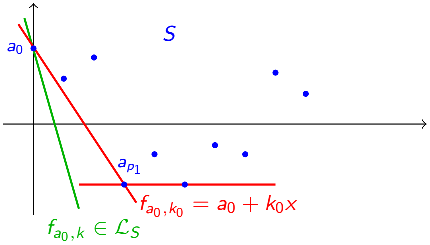

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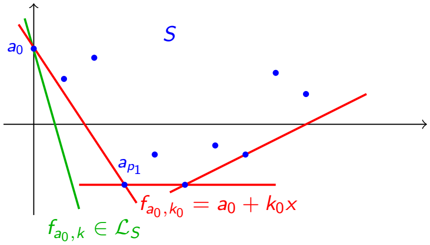

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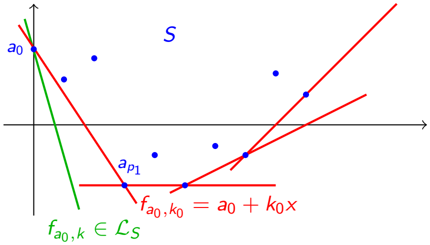

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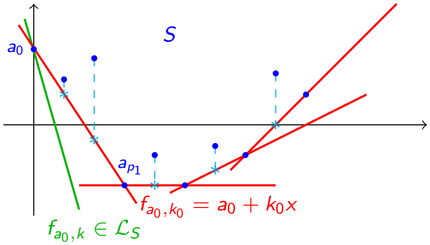

The multi-dimensional case

Theorem (Boiti-Jornet-Oliaro-Schindl)

$F(x) = \sup_{f \in \mathcal{L}_S} f(x)$ is the largest convex function $g : [0, +\infty)^d \to \mathbb{R}$ whose epigraph G_g^+ contains S, i.e. such that

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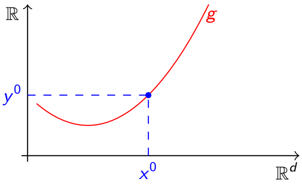

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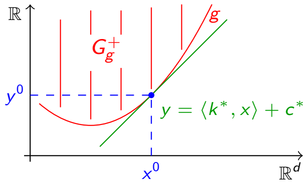

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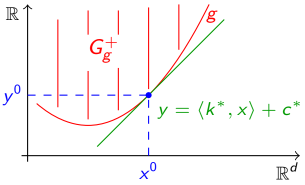

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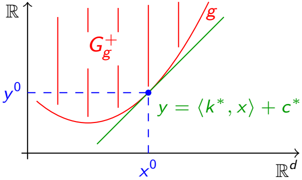

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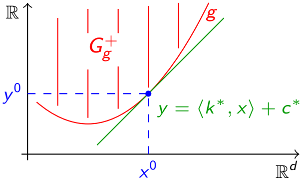

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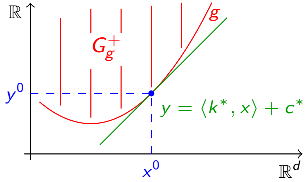

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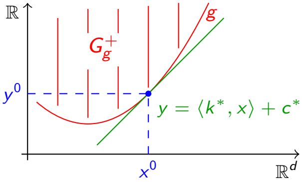

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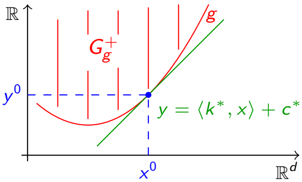

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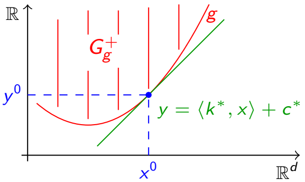

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The associated function

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The matrix weighted setting allows to treat at the same time classes in the sense of Denjoy-Carleman (estimates of the derivatives with a sequence) and in the sense of Braun, Meise and Taylor[2] (estimates of the derivatives via a weight function).

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Assumptions in the Roumieu case : (Langenbruch[3] for $\mathbf{M}^{(\lambda)} = (M_\alpha)_{\alpha\in\mathbb{N}_0^d}$)

$$\forall\lambda>0\exists\kappa\geq\lambda, A\geq 1: \quad M^{(\lambda)}_{\alpha+e_j} \leq A^{|\alpha|+1} M^{(\kappa)}_\alpha \quad \forall\alpha\in\mathbb{N}_0^d, 1\leq j\leq d \tag{4}$$

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Importance of Hermite functions

Assumptions in the Beurling case :

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The delicate question of non-triviality

Let $\mathbf{M}^{(\lambda)}=(M_\alpha)_{\alpha\in\mathbb{N}^d_0}$.

- If $\omega_{\mathbf{M}}(t)=o(t^2)$, as $t\to+\infty$,then $\mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d)$ is nontrivial (all Hermite functions are contained in this class).
  If $t^2=O(\omega_{\mathbf{M}}(t))$ then $\mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d)=\{0\}$.
- Analogously, in the Roumieu case $\omega_{\mathbf{M}}(t)=O(t^2)$ implies that $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d)$ is nontrivial but $t^2=o(\omega_{\mathbf{M}}(t))$ implies $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d)=\{0\}$.

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Characterization of inclusion relations

Theorem (Boiti-Jornet-Oliaro-Schindl)

Let $\mathcal{M} = \{(\mathbf{M}^{(\lambda)})_{\lambda>0}\}$ and $\mathcal{N} = \{(\mathbf{N}^{(\lambda)})_{\lambda>0}\}$ be two weight matrices and assume that $\mathcal{M}$ is log-convex (i.e. $\{M_\alpha^{(\lambda)}\}_\alpha$ is log-convex for all λ). Then:

- Let $\mathcal{M}$ satisfy (4)-(6) and $\mathcal{N}$ satisfy (4)-(5). Then
  (a) $\mathcal{S}_{\{\mathcal{M}\}}(\mathbb{R}^d) \subseteq \mathcal{S}_{\{\mathcal{N}\}}(\mathbb{R}^d)$ holds (with continuous inclusion);
  $\Leftrightarrow$ (b) $\mathcal{M}\{\preceq\}\mathcal{N}$, i.e. $\forall \lambda > 0 \exists \kappa > 0, C \geq 1 : \; M_\alpha^{(\lambda)} \leq C^{|\alpha|} N_\alpha^{(\kappa)} \; \forall \alpha \in \mathbb{N}_0^d$.
- Let $\mathcal{M}$ satisfy (7)-(9) and $\mathcal{N}$ satisfy (7)-(8). Then
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Application of $M_\alpha^{\mathrm{lc}} = \sup \frac{s^\alpha}{\exp \omega_{\mathbf{M}}(s)}$ in the proof

Assuming that the inclusion $\mathcal{S}_{(\mathcal{M})}(\mathbb{R}^d) \subseteq \mathcal{S}_{(\mathcal{N})}(\mathbb{R}^d)$ is continuous we first proved that for all $\ell \in \mathbb{N}$ there exist $j \in \mathbb{N}$ and $C \geq 1$ such that

$$\omega_{\mathbf{N}^{(1/\ell)}}(\ell s) \leq \log C + \omega_{\mathbf{M}^{(1/j)}}(2js), \qquad \forall s \in (0,+\infty)^d.$$

Then

$$\begin{aligned}
N_\alpha^{(1/\ell)} \geq \ (N_\alpha^{(1/\ell)})^{\mathrm{lc}} &= \sup_{t\in(0,+\infty)^d} \frac{t^\alpha}{\exp \omega_{\mathbf{N}^{(1/\ell)}}(t)} \\
&= \sup_{s\in(0,+\infty)^d} \frac{(\ell s)^\alpha}{\exp \omega_{\mathbf{N}^{(1/\ell)}}(\ell s)} \geq \frac{\ell^{|\alpha|}}{C} \sup_{s\in(0,+\infty)^d} \frac{s^\alpha}{\exp \omega_{\mathbf{M}^{(1/j)}}(2js)} \\
&= \frac{\ell^{|\alpha|}}{C} \sup_{t\in(0,+\infty)^d} \frac{\left(\frac{t}{2j}\right)^\alpha}{\exp \omega_{\mathbf{M}^{(1/j)}}(t)} = \frac{1}{C}\left(\frac{\ell}{2j}\right)^{|\alpha|} M_\alpha^{(1/j)}
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i.e. $\mathcal{M}(\preceq)\mathcal{N}$ □

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$$\omega_{\mathbf{N}^{(1/\ell)}}(\ell s) \leq \log C + \omega_{\mathbf{M}^{(1/j)}}(2js), \qquad \forall s \in (0,+\infty)^d.$$

Then

$$\begin{aligned} N_\alpha^{(1/\ell)} \geq \ (N_\alpha^{(1/\ell)})^{\mathrm{lc}} &= \sup_{t \in (0,+\infty)^d} \frac{t^\alpha}{\exp \omega_{\mathbf{N}^{(1/\ell)}}(t)} \\ &= \sup_{s \in (0,+\infty)^d} \frac{(\ell s)^\alpha}{\exp \omega_{\mathbf{N}^{(1/\ell)}}(\ell s)} \geq \frac{\ell^{|\alpha|}}{C} \sup_{s \in (0,+\infty)^d} \frac{s^\alpha}{\exp \omega_{\mathbf{M}^{(1/j)}}(2js)} \\ &= \frac{\ell^{|\alpha|}}{C} \sup_{t \in (0,+\infty)^d} \frac{\left(\frac{t}{2j}\right)^\alpha}{\exp \omega_{\mathbf{M}^{(1/j)}}(t)} = \frac{1}{C}\left(\frac{\ell}{2j}\right)^{|\alpha|} M_\alpha^{(1/j)} \end{aligned}$$

i.e. $\mathcal{M}(\preceq)\mathcal{N}$ □

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Thank you for your attention!

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Happy Birthday Pepe!!!