Mean ergodicity of multiplication operators in weighted Dirichlet spaces

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It is part of a work with Daniel Seco.

As the title of a chapter leads to studying a problem

With R. Cardeccia, K-G. Grosse-Erdmann and S. Muro, we study:

What operators satisfy,
$\exists x : T^n x \to y \neq 0 \Rightarrow \exists z : \overline{\{T^n z : n \in \mathbb{N}\}} = X$

We prove that this is true:
for M_ϕ^* in $A^2(\mathbb{D})$,
in $H^2(\mathbb{D})$ under certain restrictions.
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J. Bonet, D. Jornet and P. Sevilla-Peris, Function Spaces and Operators between them
Chapter: **Transitive and Mean Ergodic Operators**

Mean ergodicity

Let X be a Banach space.

$B(X)$ denotes the space of bounded linear operators defined on X,

X^* is the space of continuous linear functionals on X.

In our setting, X will always be a Hilbert space, and thus X^* can be identified naturally with X.

Given $T \in B(X)$, we denote its *Cesàro mean* by $M_n(T)$, which is given by

$$M_n(T)x := \frac{1}{n+1} \sum_{k=0}^{n} T^k x$$

for all $x \in X$.

Definition

A linear operator T on a Banach space X is called:

1. *Power bounded* (PB) if there is a $C > 0$ such that $\|T^n\| < C$ for all n.
2. *Cesàro bounded* (CB) if the sequence $(M_n(T))_{n \in \mathbb{N}}$ is bounded.
3. *Mean ergodic* (ME) if $M_n(T)$ converges in the strong topology of X.

$$\frac{T^n x}{n+1} = M_n x - \frac{n}{n+1} M_{n-1} x$$

Thus if T is mean ergodic, then $\frac{T^n x}{n} \to 0$, $\forall x \in X$.

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Thus if T is mean ergodic, then $\frac{T^n x}{n} \to 0$, $\forall x \in X$.

Theorem

If T is Cesaro bounded in a reflexive Banach space and $\frac{T^n}{n} x \to 0$, $\forall x \in X$, then T is mean ergodic.

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Corollary

If T is power bounded in a reflexive Banach space, then T is mean ergodic.

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Proposition

If T is Cesaro bounded, the spectrum $\sigma(T)$ is contained in the closed unit disc.

We assume $\alpha > -1$ and we consider the weighted Dirichlet space D_α consisting of all analytic functions $f(z) = \sum_{n \geq 0} a_n z^n$ defined over the unit disc $\mathbb{D}$ with

$$\|f\|_{D_\alpha}^2 := \sum_{n \geq 0} (n+1)^{1-\alpha} |a_n|^2 < \infty.$$

These are all Hilbert spaces. In particular, for $\alpha = 0, 1, 2$, we obtain the classical Dirichlet D, Hardy H^2 and Bergman A^2 spaces.

$e_n(z) = (n+1)^{-\frac{1-\alpha}{2}} z^n$ form an orthonormal basis in D_α.

The operator multiplication by z, M_z, is a forward weighted shift of the form

$$M_z e_n = \left(\frac{n+2}{n+1}\right)^{\frac{1-\alpha}{2}} e_{n+1}.$$

On the other hand, for fixed $\alpha > -1$, we can consider an equivalent norm $\|\cdot\|$ defined on D_α by

$$\|f\|^2 = |f(0)|^2 + \frac{1}{\pi}\int_{\mathbb{D}} |f'|^2(1-|z|^2)^\alpha dA.$$

Theorem

Let M_z and M_z^* act on D_α.

a) M_z and M_z^* are mean ergodic in D_α if $\alpha > 0$. (Aleman-Suciu and Bermúdez-B-Muller-Peris)

b)M_z and M_z^* are not (CB) when $\alpha = 0$. (Bermúdez-B-Muller-Peris)

Let $\alpha \geq 1$. Denote by $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Theorem

Let $\alpha \geq 1$ and $\phi \in D_\alpha$, and define M_ϕ acting on D_α. Then the following are equivalent:

(a) *M_ϕ and M_ϕ^* are (PB).*

(b) *M_ϕ and M_ϕ^* are (CB).*

(c) *$\|\phi\|_\infty \leq 1$.*

Proof.

(a)$\Rightarrow$ (b) is trivial.
(b) $\Rightarrow$ (c), since M_ϕ is (CB) then the spectrum $\sigma(M_\phi)$ is contained in closure of the unit disc. Since $\sigma(M_\phi) = \overline{\phi(\mathbb{D})}$, then $\|\phi\|_\infty \leq 1$.

(c) $\Rightarrow$ (a) reduces to using that $\|M_\phi^*\| = \|M_\phi\| = \|\phi\|_\infty$ hold. Therefore $\|M_\phi^n\| \leq 1$ for all n. □

Definition

A positive Borel measure on the open unit disc μ is called a Carleson measure for D_α if there is a constant C such that

$$\int_{\mathbb{D}} |g|^2 d\mu \leq C\|g\|_{D_\alpha}^2$$

for all $g \in D_\alpha$.

Power bounded

Theorem

Suppose $-1 < \alpha < 1$, $\phi \in M_{D_\alpha}$, $\|\phi\|_\infty \leq 1$ *and* $(\frac{|\phi'|}{1-|\phi|^2})^2(1-|z|^2)^\alpha dA$ *is a Carleson measure for* D_α. *Then* M_ϕ *and* M_ϕ^* *are (PB).*

Proof.

$$\|M_{\phi^n} f\| = |\phi^n f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |(\phi^n f)'|^2 (1-|z|^2)^\alpha dA \leq$$

We split $(\phi^n f)' = \phi^n f' + (\phi^n)' f$ to bound the norm of $M_\phi^n f$ with

$$|f(0)|^2 + \frac{2}{\pi}\int_{\mathbb{D}} |\phi^n f'|^2 (1-|z|^2)^\alpha dA + \frac{2}{\pi}\int_{\mathbb{D}} |f|^2 |(\phi^n)'|^2 (1-|z|^2)^\alpha dA.$$

□

Proof.

$$\int_{\mathbb{D}} |f|^2 |(\phi^n)'|^2 (1-|z|^2)^\alpha dA = \int_{\mathbb{D}} |f|^2 |n^2| \phi^{2n-2} |\phi'|^2 (1-|z|^2)^\alpha dA \leq$$

$$\leq \int_{\mathbb{D}} |f|^2 n^2 |\phi^{2n-2}| (1-|\phi|^2)^2 \frac{|\phi'|^2}{(1-|\phi|^2)^2} (1-|z|^2)^\alpha dA.$$

Now we can use that $x^{n-1}(1-x^2) < \frac{2}{n}$ for $0 \leq x \leq 1$, applied to $|\phi|^2$, so that the right-hand side is bounded by

$$4 \int_{\mathbb{D}} |f|^2 \frac{|\phi'|^2}{(1-|\phi|^2)^2} (1-|z|^2)^\alpha dA \leq C \|f\|^2_{D_\alpha},$$

where the last inequality comes as a direct consequence of the Carleson measure assumption. □

Recall that a *reproducing kernel Hilbert space* (RKHS) H over $\mathbb{D}$ is a Hilbert space with the property that for each $\omega \in \mathbb{D}$, there exists a unique function $k_\omega \in H$ such that

$$f(z) = \langle f, k_z \rangle.$$

Thus, from the Cauchy-Schwarz inequality, we obtain

$$|f(z)| \leq \|f\| \|k_z\|.$$

$$|f(z)|^2 \leq \begin{cases} C\frac{1}{(1-|z|^2)^\alpha}\|f\|^2_{D_\alpha} & \text{if} \quad 0 < \alpha \leq 1, \\ C\log\frac{1}{1-|z|^2}\|f\|^2_{D_\alpha} & \text{if} \quad \alpha = 0. \end{cases} \tag{1}$$

Corollary

Let $\phi \in M_{D_\alpha}$ *with* $0 \leq \alpha < 1$ *and* $\|\phi\|_\infty \leq 1$.

(a) *If* $\alpha > 0$ *and* $\int_{\mathbb{D}} (\frac{|\phi'|}{1-|\phi|^2})^2 dA < \infty$, *then* M_ϕ *and* M_ϕ^* *are (PB).*

(b) *If* $\alpha = 0$ *and* $\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} |\frac{\phi'(z)}{1-|\phi(z)|^2}|^2 dA < \infty$, *then* M_ϕ *and* M_ϕ^* *are (PB).*

Proof.

$$\frac{1}{\pi} \int_{\mathbb{D}} |f|^2 \frac{|\phi'|^2}{(1-|\phi|^2)^2} (1-|z|^2)^\alpha dA \leq C \|f\|_{D_\alpha}^2$$

Thus $(\frac{|\phi'|}{1-|\phi|^2})^2 (1-|z|^2)^\alpha dA$ is a Carleson measure for D_α. □

Corollary

Let $\phi \in M_{D_\alpha}$ for $0 < \alpha < 1$. If $\phi : \mathbb{D} \to \mathbb{D}$ is univalent and $\phi(\mathbb{D})$ has finite hyperbolic area, then M_ϕ and M_ϕ^* are (PB).

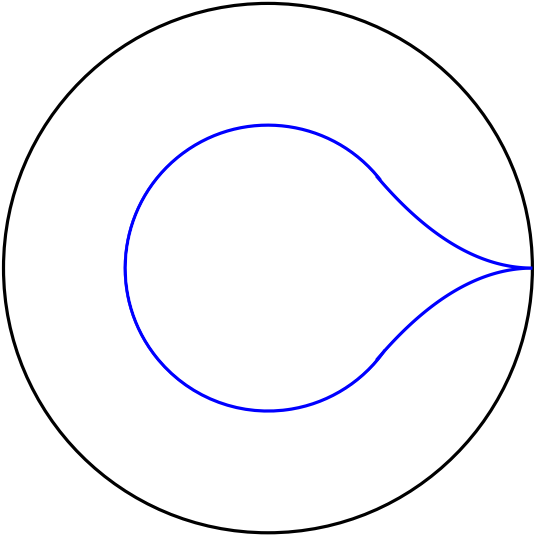
Proof.

If $G := \phi(\mathbb{D})$ has finite hyperbolic area, then $\int_G \frac{1}{(1-|z|^2)^2} dA < \infty$.
Thus

$$\int_{\mathbb{D}} \left(\frac{|\phi'|}{1 - |\phi|^2} \right)^2 dA = \int_G \frac{1}{(1 - |z|^2)^2} dA < \infty$$

□

In particular, if $\phi : \mathbb{D} \to \mathbb{D}$ is univalent and $\phi(\mathbb{D}) \subset \mathbb{D}$ has boundary contacting the circle only at the point 1 and furthermore, that near 1, G lies between the graph of $y = (1-x)^2$ and its reflection in the x-axis.



Cesàro boundedness

Theorem

Let $\phi \in M_{D_\alpha}$ *with* $-1 < \alpha < 1$, $\|\phi\|_\infty \leq 1$ *and* $|\frac{\phi'(z)}{1-\phi(z)}|^2(1-|z|^2)^\alpha dA$ *be a Carleson measure for* D_α. *Then* M_ϕ *and* M_ϕ^* *are (CB).*

Corollary

Suppose $\phi \in M_{D_\alpha}$ *with* $0 \leq \alpha < 1$ *and* $\|\phi\|_\infty \leq 1$. *Suppose any of the following hold:*

(a) $\alpha > 0$ *and* $\int_{\mathbb{D}} |\frac{\phi'}{1-\phi}|^2 dA < \infty$.

(b) $\alpha = 0$ *and* $\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} |\frac{\phi'(z)}{1-\phi(z)}|^2 dA < \infty$.

Then M_ϕ *and* M_ϕ^* *are (CB) (in the corresponding* D_α*).*

In particular that M_z is Cesàro bounded in D_α for $\alpha > 0$. Indeed, a well known inequality (see Aleman-Persson) assures that

$$\int_{\mathbb{D}} \frac{|f(z)|^2}{|1-z|^2}(1-|z|^2)^\alpha dA \leq C\|f\|_{D_\alpha}^2.$$

Thus $\frac{(1-|z|^2)^\alpha}{|1-z|^2} dA$ is a Carleson measure for D_α.

Example

For $z \in \mathbb{D}$ and $\theta \neq 0$ define ϕ by

$$\phi(z) = e^{i\theta} \cdot \frac{1-z}{2} \cdot e^{-\frac{1+z}{1-z}}.$$

Then M_ϕ is Cesàro bounded when acting on the Dirichlet space.

Proof.

We need show that $\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} |\frac{\phi'(z)}{1-\phi(z)}|^2 dA$ is bounded.
We can deduce from Galanopoulos, Girela, M. J. Martín that they prove that $\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} |\phi'(z)|^2 dA$ is finite. Then we can use that $|1-\phi(z)| > c > 0$.

□

Mean ergodicity

Theorem

Let $\alpha \in (-1,1)$ and $\phi \in M_{D_\alpha}$. Then M_ϕ is (ME) if and only if it is (CB).

Proof.

($\Rightarrow$) Any mean ergodic operator is Cesàro bounded.
($\Leftarrow$) Since M_ϕ is Cesàro bounded and the space is reflexive, it is sufficient to establish that for all $f \in D_\alpha$, we have $\frac{\|M_\phi^n f\|}{n} \to 0$ (as $n \to \infty$). The (CB) property implies already that $\|\phi\|_\infty \leq 1$, and if $|\phi(z)| = 1$ at some $z \in \mathbb{D}$ then ϕ must be constant and $\left\|\frac{M_\phi^n f}{n}\right\|_{D_\alpha} \to 0$. □

Proof.

We can otherwise assume that $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. We want to study the quantity

$$\left\| \frac{M_\phi^n f}{n} \right\|_{D_\alpha}^2 = \left| \frac{\phi^n(0) f(0)}{n} \right|^2 + \frac{1}{n^2 \pi} \int_{\mathbb{D}} |(\phi^n f)'|^2 (1-|z|^2)^\alpha dA.$$

$$\frac{2}{n^2\pi} \int_{\mathbb{D}} |\phi^n f'|^2 (1-|z|^2)^\alpha dA + \frac{2}{\pi} \int_{\mathbb{D}} |f|^2 |\phi^{n-1}|^2 |\phi'|^2 (1-|z|^2)^\alpha dA.$$

The first integral can be bounded by $\frac{2\|f\|_{D_\alpha}^2}{n^2}$ using that $|\phi| < 1$.
For second integral we use Lebesgue dominated convergence Theorem. It can be applied since its integrand is vanishing (because $|\phi| < 1$) and bounded above by $|f(z)|^2 |\phi'|^2 (1-|z|^2)^\alpha$, and use that $|\phi'|^2 (1-|z|^2)^\alpha$is a Carleson measure, since ϕ is a multiplier. □

Corollary

For $\alpha > 0$, M_z *is (ME) in* D_α.

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Theorem

Let $0 \leq \alpha < 1$ *be and* $\phi \in M_{D_\alpha}$. *When* $\alpha \in (0,1)$ *assume moreover that* $\int_{\mathbb{D}} |\phi'(z)|^2 dA < \infty$ *(that is,* ϕ *is in the classical Dirichlet space). When* $\alpha = 0$, *assume* $\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} |\phi'(z)|^2 dA < \infty$. *Then* M_ϕ^* *is (ME) if and only if it is (CB).*

Corollary

Let $0 < \alpha < 1$, $\phi \in M_{D_\alpha}$ *and* $\|\phi\|_\infty \leq 1$. *If* $\int_{\mathbb{D}} |\frac{\phi'(z)}{1-\phi(z)}|^2 dA < \infty$, *then* M_ϕ *and* M_ϕ^* *are (ME).*

For the classical Dirichlet space we obtain the following:

Corollary

Let $\phi \in M_{D_0}$ *and* $\|\phi\|_\infty \leq 1$. *If* $\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} |\frac{\phi'(z)}{1-\phi(z)}|^2 dA < \infty$, *then* M_ϕ *and* M_ϕ^* *are (ME).*

Example

Let $\phi(z) = e^{i\theta}(\frac{1-z}{2})^k$ for some $\theta \neq 0$ and $k \in \mathbb{N}$, $k \geq 1$. Then M_ϕ and M_ϕ^* acting on D_0 are (ME) but not (PB).

Proof.

Notice $\|\phi\|_\infty \leq 1$. Moreover, it is well known that $\int_{\mathbb{D}} |1-z|^{2\beta} \log \frac{2}{1-|z|^2} dA$ is bounded if $\beta > -1$ Galanopoulos-Girela-Martín. If we also use that $|1-\phi(z)| > c > 0 \; \forall z \in \mathbb{D}$, we have

$$\int_{\mathbb{D}} \log \frac{1}{1-|z|^2} \left| \frac{\phi'(z)}{1-\phi(z)} \right|^2 dA < \infty.$$

□

Proof.

$$(\frac{1-z}{2})^m = 2^{-m}\sum_{k=1}^{m}\binom{m}{k}(-z)^k.$$

$$\|(\frac{1-z}{2})^m\|^2 = 2^{-2m}\sum_{k=1}^{m}\binom{m}{k}^2(k+1) \geq 2^{-2m}m\sum_{k=1}^{m}\binom{m}{k}\binom{m-1}{m-k}.$$

By the Chu-Vandermonde identity gives us that

$$\|(\frac{1-z}{2})^m\|^2 \geq 2^{-2m}m\binom{2m-1}{m}.$$

An application of the Stirling approximation formula yields

$$\|(\frac{1-z}{2})^m\|^2 \approx 2^{-2m}m\binom{2m-1}{m} = \frac{(2m-1)!}{2^{2m}((m-1)!)^2} \approx \sqrt{m}.$$

Now, since $\|M_\phi^n\| \geq \|\phi^n\|$ and as far as $\|(\frac{1-z}{2})^{kn}\|^2 \geq C\sqrt{n}$, M_ϕ can not be power bounded. □

THANK YOU SO MUCH