Endpoint estimates for Rubio de Francia operators

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Workshop on Functional Analysis on the Occasion of the 70th Birthday of José Bonet

(17-19 mayo, 2025)

Dedicated to

José Bonet's head

For his 70^{th} birthday

Abstract

General situation

$$T : L^p \longrightarrow L^p, \quad \forall p \in (p_0, p_1)$$

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What can we say at the endpoints p_0 and p_1?

This is a joint work with Teresa Luque. and Laura Sánchez-Pascuala.

Many answers

Depending on the information

a) Strong type

$$T : L^{p_0} \longrightarrow L^{p_0}$$

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$$T : L^{p_0} \longrightarrow L^{p_0}$$

b) Weak type

$$T : L^{p_0} \longrightarrow L^{p_0,\infty}$$

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$$T : L^{p_0} \longrightarrow L^{p_0}$$

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c) Restricted weak type

$$T : L^{p_0,1} \longrightarrow L^{p_0,\infty}$$

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d) There exists $X \subset L^{p_0}$

$$T : X \longrightarrow L^{p_0}$$

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d) There exists $X \subset L^{p_0}$

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e) It is an open question.

Function spaces

Function spaces

Weighted Lebesgue spaces

$$L^p(v) = \left\{ f : \|f\|_{L^p(v)} = \left(\int |f(x)|^p v(x) dx \right)^{1/p} < \infty \right\}$$

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Lorentz spaces

$$L^{p,q}(v) = \left\{ f : \|f\|_{L^{p,q}} = \left(\int_0^\infty f_v^*(t)^q t^{q/p-1} dt \right)^{1/q} < \infty \right\}$$

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Logarithmic spaces

$$L(\log L)^m = \left\{ f : \|f\|_{L(\log L)^m} = \int_0^\infty f_v^*(t) \left(1 + \log_+ \frac{1}{t} \right)^m dt < \infty \right\}$$

Rubio de Francia operators

The Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy.$$

Rubio de Francia operators

The Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

Theorem (Muckehoupt 1972)

$$M : L^p(v) \longrightarrow L^p(v) \quad \Longleftrightarrow \quad v \in A_p.$$

$$M : L^1(v) \longrightarrow L^{1,\infty}(v) \quad \Longleftrightarrow \quad v \in A_1.$$

B. Muckenhoupt, Trans. Amer. Math. Soc. (1972).

Rubio de Francia operators

Definition

An operator T is said to be a Rubio de Francia operator if, for every $v \in A_p$,

$$T : L^p(v) \longrightarrow L^p(v)$$

is bounded.

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Examples

1. M, the Hardy-Littlewood maximal operator.
2. H, the Hilbert transform.
3. R_j the Riesz transform.
4. Any Calderón-Zygmund operator.
5. etc.

Rubio de Francia's Extrapolation Theorem

Theorem

Let T be an operator such that, for some $p_0 \geq 1$ and every $v \in A_{p_0}$,

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v)$$

is bounded. Then, for every $p > 1$ and every $v \in A_p$,

$$T : L^p(v) \longrightarrow L^p(v)$$

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J. L. Rubio de Francia, Amer. J. Math. **106** (1984).

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Rubio de Francia's Extrapolation Theorem

Fundamental remark

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v), \qquad \forall\, v \in A_{p_0}.$$

$$\not\Downarrow$$

$$T : L^{1}(v) \longrightarrow L^{1,\infty}(v), \qquad \forall\, v \in A_{1}.$$

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$$T : L^1(v) \longrightarrow L^{1,\infty}(v), \qquad \forall\, v \in A_1.$$

Example: $M \circ M$.

Quantitative version of Rubio de Francia's Extrapolation Theorem

Theorem

Assume that, for some $1 \le p_0 < \infty$, and for all $v \in A_{p_0}$

$$\left(\int_{\mathbb{R}^n} |Tf|^{p_0} v \right)^{1/p_0} \le N(\|v\|_{A_{p_0}}) \left(\int_{\mathbb{R}^n} f^{p_0} v \right)^{1/p_0}.$$

Then, for all $1 < p < \infty$ and all $v \in A_p$,

$$\left(\int_{\mathbb{R}^n} |Tf|^{p} v \right)^{1/p} \le K(v) \left(\int_{\mathbb{R}^n} f^{p} v \right)^{1/p},$$

where:

Quantitative of Rubio de Francia's Extrapolation Theorem

Sharp estimate

$$K(v) \lesssim N\left(\frac{\|v\|_{A_p}^{\max\left(1,\frac{p_0-1}{p-1}\right)}}{p-1}\right).$$

O. Dragičević, L. Grafakos, M. C. Pereyra and S. Petermichl, Publ. Mat. **49** (2005).

Duoandikoetxea, J. Funct. Anal. **260** (2011).

Endpoint estimates near L^1

Theorem

If

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v), \qquad \|v\|^s_{A_{p_0}},$$

then, for every $1 < p < p_0$,

$$T : L^p(v) \longrightarrow L^p(v), \qquad \forall\, v \in A_1,$$

with

$$||T|| \lesssim \frac{1}{(p-1)^{s(p_0-1)}}, \qquad \text{as } p \to 1^+.$$

Yano's extrapolation Theorem, 1951

If ν is a finite measure and T is a sublinear operator T such that

$$T : L^p(\mu) \longrightarrow L^p(\nu), \qquad (p-1)^{-m}, m > 0,$$

then

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\nu).$$

Optimality of Yano's extrapolation Theorem

$$M : L^p((0,1)) \longrightarrow L^p((0,1)), \qquad \frac{1}{p-1},$$

and hence

$$M : L\log L((0,1)) \longrightarrow L^1((0,1)),$$

Endpoint estimates near L^1

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and hence

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but, in fact,

$$Mf \in L^1((0,1)) \quad \Longleftrightarrow \quad f \in L(\log L)((0,1)).$$

Corollary

Let $1 < p_0 < \infty$, $s > 0$ and let T be a sublinear operator such that

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v), \qquad \|v\|_{A_{p_0}}^s.$$

Then,

$$T : L(\log L)^{s(p_0-1)} \longrightarrow L^1_{loc}$$

is also bounded.

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Then,

$$T : L(\log L)^{s(p_0-1)}(u) \longrightarrow L^1_{loc}(u)$$

is also bounded, for every $u \in A_1$.

Endpoint estimates near L^1

Theorem

If a sublinear operator T satisfies that

$$T : L^{p,1}(\mu) \longrightarrow L^p(\nu), \qquad (p-1)^{-m}, m > 0,$$

then

$$T : L(\log L)^m(\mu) \longrightarrow \Lambda^1_\nu(w) \subset L^1_{loc}(\nu)$$

Remark on Lorentz spaces

$$L^{p,1} \subset L^p \subset L^{p,\infty}$$

MJC, J. Funct. Anal. 174 (2000).

The convergence of Fourier series for functions in $L^1(0,1)$

Main motivation: Convergence of the Fourier series

In 1967 Hunt proved that

$$\|S^*\chi_E\|_{L^{p,\infty}(0,1)} \lesssim \frac{1}{p-1}|E|^{1/p},$$

where

$$S^*f(x) = \sup_N S_N f(x), \qquad S_N f(x) = \sum_{|k|\leq N} \hat{f}(k) e^{2\pi i n x}$$

Then

$$S^* : L^{p,1}(0,1) \longrightarrow L^1(0,1), \qquad \|S^*\| \lesssim \frac{1}{(p-1)^2},$$

The convergence of Fourier series for functions in $L^1(0,1)$

Main motivation: Convergence of the Fourier series

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In 1996, Antonov used Hunt's estimate to obtain the convergence for functions in $L \log L \log \log \log L(0,1)$ where

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Extensions of Yano's theorem

- Different endpoints

$$T : L^p \longrightarrow L^p \qquad \|T\| \lesssim p^\alpha, \quad \forall\, p < \infty,$$

$$T : L^p \longrightarrow L^p \qquad \|T\| \lesssim \frac{1}{(p_1 - p)^\alpha}, \quad \forall\, p < p_1 < \infty,$$

$$T : L^p \longrightarrow L^p \qquad \|T\| \lesssim \frac{1}{(p - p_0)^\alpha}, \quad \forall\, p > p_0 > 1.$$

Extensions of Yano's theorem

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$$
\begin{aligned}
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\end{aligned}
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- Different norm behaviour

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T : L^p \longrightarrow L^p \qquad \|T\| \lesssim \log\left(\frac{1}{p-1}\right).
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- Different types of spaces in the hypotheses.

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$$T : L^p \longrightarrow L^p \qquad \|T\| \lesssim \log\left(\frac{1}{p-1}\right).$$

- Different types of spaces in the hypotheses.
- An abstract theory (M. Milman and B. Jawerth).

Rubio de Francia operator

Let T be a RF operator such that

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v), \qquad \|v\|^{s}_{A_{p_0}}.$$

Then

$$\begin{aligned} T &: L^{p_0}(v) \longrightarrow L^{p_0,\infty}(v), &\qquad \|v\|^{\sigma}_{A_{p_0}} \\ T &: L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v), &\qquad \|v\|^{r}_{A_{p_0}} \end{aligned}$$

with

$$r \leq \sigma \leq s,$$

Examples

$$\|M\|_{L^2(v)\to L^2(v)} \lesssim \|v\|_{A_2}, \qquad \|M\|_{L^2(v)\to L^{2,\infty}(v)} \lesssim \|v\|_{A_2}^{1/2},$$

$$\|M\|_{L^{2,1}(v)\to L^{2,\infty}(v)} \lesssim \|v\|_{A_2}^{1/2},$$

Endpoint estimates near L^1

Theorem (Yano's type result)

If a sublinear operator T satisfies that

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \qquad (p-1)^{-m}, m > 0,$$

then

$$T : L(\log L)^m(\log_3 L)(\mu) \longrightarrow L^1_{loc}(\nu),$$

where $\log_3 = \log\log\log$.

MJC and J. Martín, Rev. Mat. Iberoamericana 20 (2004).

Summarize:

$T: L^{p_0}(w) \to L^{p_0}(w)$ $\\|T\\| \lesssim \\|w\\|_{A_{p_0}}^{s}$	$T: L^{p_0}(w) \to L^{p_0,\infty}(w)$ $\\|T\\| \lesssim \\|w\\|_{A_{p_0}}^{\sigma}$	$T: L^{p_0,1}(w) \to L^{p_0,\infty}(w)$ $\\|T\\| \lesssim \\|w\\|_{A_{p_0}}^{r}$	$T: L^{p_0,\infty}(w) \to L^{p_0,\infty}(w)$ $\\|T\\| \lesssim \\|w\\|_{A_{p_0}}^{\alpha}$
$T: L^{p}(u) \to L^{p}(u)$ $\\|T\\| \lesssim \frac{1}{(p-1)^{s(p_0-1)}}$	$T: L^{p}(u) \to L^{p,\infty}(u)$ $\\|T\\| \lesssim \frac{1}{(p-1)^{\sigma(p_0-1)}}$	$T: L^{p,p/p_0}(u) \to L^{p,\infty}(u)$ $\\|T\\| \lesssim \frac{1}{(p-1)^{r(p_0-1)}}$	$T: L^{p,\infty}(u) \to L^{p,\infty}(u)$ $\\|T\\| \lesssim \frac{1}{(p-1)^{\alpha(p_0-1)}}$
$L(\log L)^{s(p_0-1)}(u)$	$L(\log L)^{\sigma(p_0-1)} \log_3 L(u)$	$L(\log L)^{r(p_0-1)+b} \log_3 L(u)$ $b = \min(1, p_0 - 1)$ (ε, δ)-atomic case: $b = 0$	$[L(\log L)^{\alpha(p_0-1)-1} \log_3 L(u)]_1$

M.C. and C. Domingo-Salazar, TAMS, 2016

How can we prove that Yano's type results?

The ODR technique

1. O: Optimization
2. D: Decomposition
3. R: Reconstruction

Optimization of Hypotheses.

To obtain an endpoint estimate for a *subset* $\mathcal{O}$ of $L^1(\mu)$

$$\mathcal{O} = \left\{ f \in L^1(\mu) \cap L^\infty(\mu) : \ \|f\|_{L^\infty(\mu)} \leq 1 \right\}.$$

$$\|Tf\|_{L^1(\nu)} \lesssim \|Tf\|_{L^p(\nu)} \lesssim \frac{\|f\|_{L^p(\mu)}}{(p-1)^\alpha} \lesssim \frac{\|f\|_{L^1(\mu)}^{1/p}}{(p-1)^\alpha}.$$

Hence, taking the infimum over $1 < p \leq 2$ on the right hand side, we obtain

$$\|Tf\|_{L^1(\nu)} \lesssim \|f\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f\|_{L^1(\mu)}} \right)^\alpha, \qquad \forall f \in \mathcal{O}.$$

Decomposition.

To find a decomposition of a function $f \in L^1(\mu)$ as a linear combination of functions in $\mathcal{O}$

1) Dyadic decomposition:

$$f = \sum_{i \in \mathbb{Z}} 2^i f_i \quad \text{where} \quad f_i = \frac{f\chi_{\{2^{i-1} \leq |f| < 2^i\}}}{2^i}.$$

2) k-dyadic decomposition:

$$f = \sum_{n \geq 0} d_n f_n, \qquad d_n = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{k\text{-times}} \text{ and } f_n = \frac{f\chi_{\{d_{n-1} \leq |f| < d_n\}}}{d_n}$$

with $d_{-1} = 0$. If $k = 2$, we call *super-dyadic* decomposition.

Reconstruction.

To gather the obtained information.

$$\|Tf\|_{L^1(\nu)} \lesssim \sum_{i\in\mathbb{Z}} 2^i \|Tf_i\|_{L^1(\nu)} \lesssim \sum_{i\in\mathbb{Z}} 2^i \|f_i\|_{L^1(\mu)} \left(\log_1 \frac{1}{\|f_i\|_{L^1(\mu)}}\right)^{\alpha}$$

$$\lesssim \int_0^\infty \lambda_f^\mu(s) \left(\log_1 \frac{1}{\lambda_f^\mu(s)}\right)^{\alpha} ds \lesssim \|f\|_{L(\log L)^\alpha(\mu)},$$

Reconstruction.

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$$\lesssim \int_0^\infty \lambda_f^\mu(s) \left(\log_1 \frac{1}{\lambda_f^\mu(s)}\right)^\alpha ds \lesssim \|f\|_{L(\log L)^\alpha(\mu)},$$

What if the final space is not Banach?.

$$\|Tf\|_{L^{1,\infty}(\nu)} \lesssim \sum_{i\in\mathbb{Z}} 2^i \log i \,\|Tf_i\|_{L^{1,\infty}(\nu)} \lesssim \cdots \lesssim \|f\|_{L(\log L)^\alpha(\log\log L)(\mu)}.$$

Reconstruction.

Can we improve the previous estimate?.

The idea (original inspired by the work of Antonov) was to consider the super-dyadic decomposition

$$\begin{aligned} \|Tf\|_{L^{1,\infty}(\nu)} &\lesssim \sum_{i\in\mathbb{Z}} 2^{2^i} \log i \, \|Tf_i\|_{L^{1,\infty}(\nu)} \\ &\lesssim \cdots \lesssim \|f\|_{L(\log L)^\alpha(\log\log\log L)(\mu)} \,. \end{aligned}$$

Multilinear operators (Bilinear)

Definition

An operator

$$B : X \times Y \longrightarrow Z$$

is said to be bilinear if it is linear in each variable. And, we say it is bounded if

$$||B(x,y)||_Z \lesssim ||x||_X ||y||_Y$$

Examples

Bilinear Hilbert trasform

$$B(f,g)(x) = p.v. \int_{\mathbb{R}} \frac{f(x-t)g(x+t)}{t} dt.$$

Bilinear Yano's extrapolation

Goal L. Sánchez-Pascuala, PhD Thesis (Advisors: MJC and T. Luque). May 6, 2025

Let B be a bilinear operator such that,

$$B : L^{p_1,r_1} \times L^{p_2,r_2} \longrightarrow L^{p,s},$$

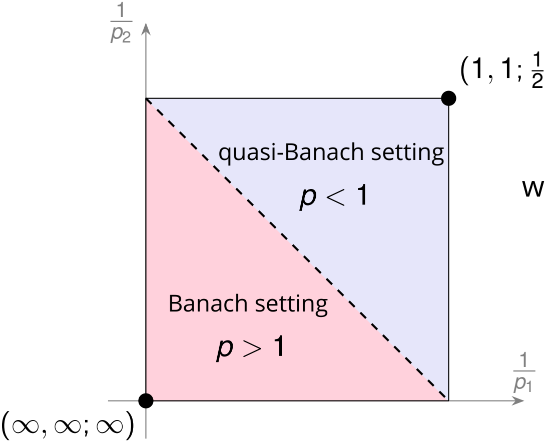
with

$$\|B\| \lesssim \varphi_1\left(\frac{1}{|p_1 - q_1|}\right)\varphi_2\left(\frac{1}{|p_2 - q_2|}\right),$$

where $r_1 \in \{1, p_1, \infty\}$, $r_2 \in \{1, p_2, \infty\}$, $s \in \{p, \infty\}$.

M. C., T. Luque and L. Sánchez-Pascuala, Linear and Bilinear Yano's extrapolation for Lorentz spaces. In preparation.

Bilinear Yano's extrapolation

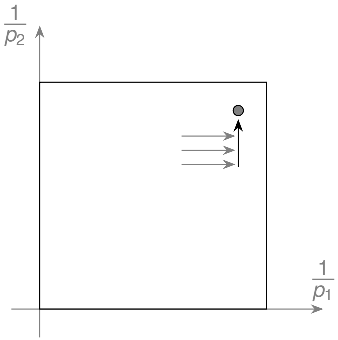


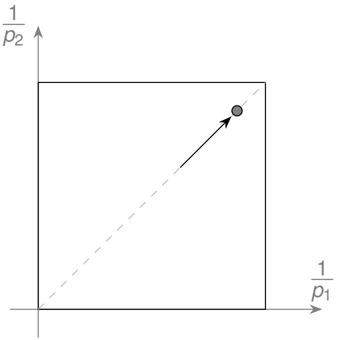
$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p,$$

where $(p_1, p_2; p)$ satisfies

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Bilinear Yano's extrapolation

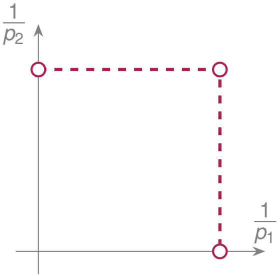


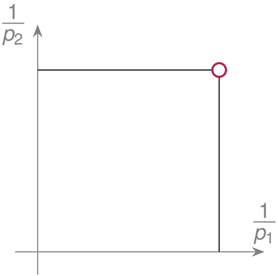


Bilinear Yano's extrapolation

$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p,$$

$$\|B\| \lesssim \frac{1}{(p_1 - 1)^{\alpha_1}(p_2 - 1)^{\alpha_2}}, \qquad \|B\| \lesssim \frac{1}{\left(p - \frac{1}{2}\right)^{\alpha}}.$$





Theorem

Let B be a bilinear operator such that, f

$$B : L^{p_1,1} \times L^{p_2,1} \longrightarrow L^{p,\infty}, \qquad \|B\| \lesssim \frac{1}{(p_1-1)^{\alpha_1}(p_2-1)^{\alpha_2}},$$

then, for every $\varepsilon > 0$,

$$B : L(\log L)^{\alpha_1}(\log\log L)^{1+\varepsilon} \times L(\log L)^{\alpha_2}(\log\log L)^{1+\varepsilon} \longrightarrow L^{1/2,\infty}_{loc},$$

Definition

The bilinear Bochner-Riesz operator with index $\alpha \geq 0$

$$\mathcal{B}^{\alpha}(f,g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1-|\xi|^2-|\eta|^2)_+^{\alpha}\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

The bilinear Bochner-Riesz

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- For $\alpha > n - \frac{1}{2}$, $\mathcal{B}^{\alpha}$ is bounded for every $1 \leq p_1, p_2 \leq \infty$,
- $\alpha = n - \frac{1}{2}$ is the critical index.

The bilinear Bochner-Riesz operator

Easy case

If $\alpha > n - \frac{1}{2}$, then

$$\mathcal{B}^\alpha(f, g) \lesssim Mf(x)Mg(x).$$

In this case:

For which $(p_1, p_2; p)$, it holds that

$$\mathcal{B}^\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $p_1 > 1$ or $p_2 > 1$. And,

$$\mathcal{B}^\alpha : L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \longrightarrow L^{1/2,\infty}(\mathbb{R}^n),$$

Theorem

For every $v_1, v_2 \in A_2$,

$$\mathcal{B}^{n-\frac{1}{2}} : L^2(v_1) \times L^2(v_2) \longrightarrow L^1(v),$$

with

$$\left\| \mathcal{B}^{n-\frac{1}{2}} \right\| \lesssim [v_1]_{A_2}^3 [v_2]_{A_2}^3.$$

K. Jotsaroop, S. Shrivastava, and K. Shuin. Weighted estimates for bilinear Bochner-Riesz means at the critical index. Potential Anal., (2021).

Bilinear Rubio de Francia

Theorem (L. Grafakos and J. Martell, (2004))

If, for some $(r_1, r_2; r)$ and, for every $v_j \in A_{r_j}$,

$$B : L^{r_1}(v_1) \times L^{r_2}(v_2) \longrightarrow L^r(v), \quad \|B\| \lesssim \psi\left([v_1]_{A_{r_1}}, [v_2]_{A_{r_2}}\right),$$

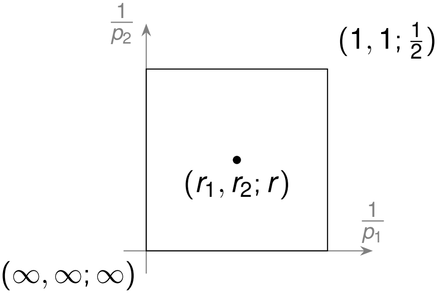
where ψ is an increasing function, then, for $(p_1, p_2; p)$ with $1 < p_1, p_2 < \infty$ and, for $v_j \in A_{p_j}$,

$$B : L^{p_1}(v_1) \times L^{p_2}(v_2) \longrightarrow L^p(v),$$

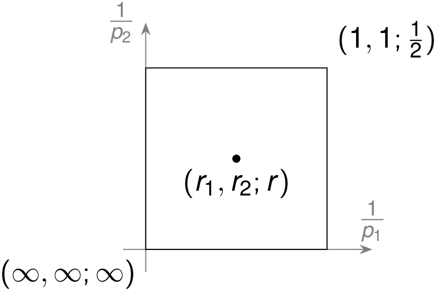
with the following operator norm: If $1 < p_1 < r_1$, $1 < p_2 < r_2$ and $v_1, v_2 \in A_1$,

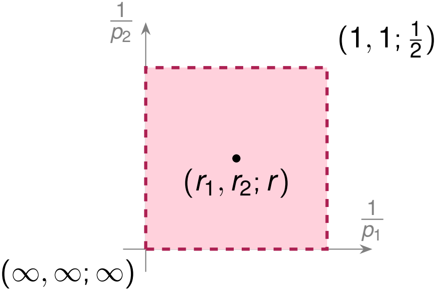
$$\|B\| \lesssim \psi\left(\left(\frac{1}{p_1 - 1}\right)^{r_1 - p_1} [v_1]_{A_1}^{1 + r_1 - p_1}, \left(\frac{1}{p_2 - 1}\right)^{r_2 - p_2} [v_2]_{A_1}^{1 + r_2 - p_2}\right).$$

Bilinear Rubio de Francia Theorem

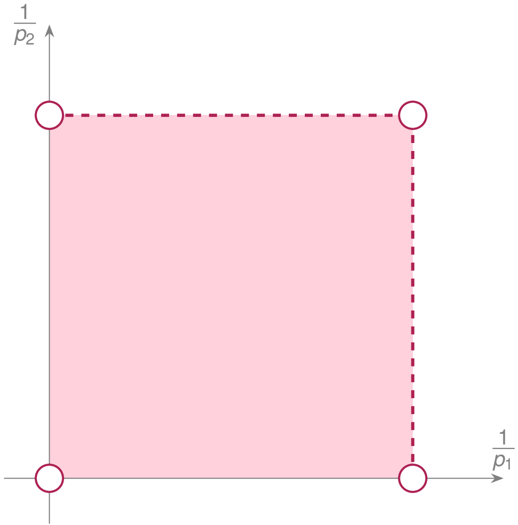


Bilinear Rubio de Francia Theorem





The bilinear Bochner-Riesz operator at the critical index



Corollary

For every $1 < p_1 \leq 2$ and $1 < p_2 \leq 2$,

$$B : L^{p_1} \times L^{p_2} \longrightarrow L^p,$$

with

$$\left\| \mathcal{B}^{n-\frac{1}{2}} \right\| \lesssim \left(\frac{1}{p_1 - 1} \right)^3 \left(\frac{1}{p_2 - 1} \right)^3.$$

The bilinear Bochner-Riesz operator of index $\alpha = n - \frac{1}{2}$

Theorem: the one weight case $(1, 1; 1/2)$

For every $\varepsilon > 0$, $v \in A_1$, and

$$D(v) := L(\log L)^3(\log_2 L)^{(1+\varepsilon)}(v),$$

we have that

$$\mathcal{B}^{n-\frac{1}{2}} : D(v) \times D(v) \longrightarrow R \subset L^{1/2,\infty}_{loc}(v),$$

where

$$||f||_R = \sup_{t>0} \frac{t^2 f_v^*(t)}{(\log_1 t)^6}.$$

The bilinear Bochner-Riesz operator of index $\alpha = n - \frac{1}{2}$

Theorem: the two weight case $(1, 1; 1/2)$

For every $\varepsilon > 0$, $v_1, v_2 \in A_1$, $v = v_1^{1/2} v_2^{1/2}$, and

$$D(v_j) := L(\log L)^6 (\log_2 L)^{(1+\varepsilon)}(v_j),$$

we have that

$$\mathcal{B}^{n-\frac{1}{2}} : D(v_1) \times D(v_2) \longrightarrow R \subset L^{1/2,\infty}_{loc}(v),$$

where

$$||f||_R = \sup_{t>0} \frac{t^2 f_v^*(t)}{(\log_1 t)^6}.$$

The end

Thanks for your attention!