Recent results on the finite Hilbert transform on $(-1, 1)$

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Universitat Politècnica de València

Authorship

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Curbera, Okada, Ricker $\equiv$ COR

Outline

Origins of the finite Hilbert transform

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> *"For a two dimensional flow of an ideal fluid past a thin airfoil there arise two types of problems: the "thickness" problem and the "lifting" problem; they lead to two different types of boundary value problem for the complex velocity $w = u - iv$ in the complex plane of flow, $z = x + iy$."*
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The second problem lead to the *"integral equation of the lifting problem"*, also known as the airfoil equation (Tragfläche), namely

$$v_0(x) = \frac{1}{\pi}\int_{\alpha_1}^{\alpha_2} \frac{u_0(s)}{s-x}\, ds,$$

where $v_0(x) := v(x,+0)$ and $u_0(x) := u(x,+0)$.

Study of the finite Hilbert transform

- The early treatment of the lifting problem and its **inversion**:

Betz in 1919, Carleman in 1922, Birnbaum in 1923, Munk in 1924, Glauert in 1924-25, von Kármán in 1930, Hamel in 1937, Söhngen in 1939.

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Die tragende Wirbelfläche als Hilfsmittel zur Behandlung des ebenen Problems der Tragflügeltheorie.

Von **W. BIRNBAUM** in Berlin.[1]

(Auf Grund von Rechnungen von W. Ackermann.)

Im folgenden wird gezeigt, wie sich die Prandtlsche Theorie des »tragenden Wirbels«[2]) im ebenen Problem der Tragflügeltheorie dazu verwenden läßt, die Verteilung des Auftriebs nach der Tiefe des Tragdecks in Rechnung zu ziehen und die Abhängigkeit dieser Verteilung von der Profilform zu finden. Dazu nehme ich den Uebergang von der einfachen tragenden Linie zur tragenden (unendlich dünnen) Fläche vor, rechne also mit der nächsthöheren Näherung (die dritte Näherung wären Flächenwirbel auf dem Rande

Study of the finite Hilbert transform

- Problems in Elasticity Theory led to the study of one-dimensional Singular Integral Operators by the Soviet School:

Duduchava, Gakhov, Gohberg, Khvedelidze, Krupnik, Mikhlin, Muskhelishvili, Prössdorf, Vekua.

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• The L^p-theory for the FHT was studied by:

Tricomi in 1951, Söhngen in 1954, Widom in 1960, Jörgens in 1970.

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• Nowadays: a celebrated 1991 paper by Gelfand and Graev has shown its application to **image reconstruction in Tomography**.

Outline

The finite Hilbert transform, FHT

The finite Hilbert transform is defined, for $f \in L^1(-1, 1)$, via the principal value integral:

$$Tf(t) := \text{p.v.}\frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{x-t}\, dx, \quad t \in (-1, 1).$$

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It is related to the solution of the **airfoil equation**: given g find all functions f which satisfy

$$g(t) = \text{p.v.}\frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{x-t}\,dx, \quad t \in (-1,1).$$

Boundedness of the FHT in L^p

- Theorem (M. Riesz): For H the Hilbert transform in $\mathbb{R}$:

$$H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \iff 1 < p < \infty.$$

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- The result extends to the FHT:

$$T\colon L^p(-1,1) \to L^p(-1,1) \iff 1 < p < \infty.$$

This allows to study the L^p-theory for the FHT.

Outline

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$$T\colon L^p(-1,1) \to L^p(-1,1):$$

There were several cases with empty sets:

	$\sigma_{\mathrm{pt}}(T_p)$	$\sigma_{\mathrm{r}}(T_p)$	$\sigma_{\mathrm{c}}(T_p)$
$1 < p < 2$		$\varnothing$	
$p = 2$	$\varnothing$	$\varnothing$	
$2 < p < \infty$	$\varnothing$		

Beyond L^p-spaces: Why?

Theorem (COR, 2021)

Let X be a (rearrangement invariant) space such that

$$T_X = T\colon X \to X$$

boundedly. The following ***alternative*** *holds.*

(a) *The point spectrum:* $\sigma_{\mathrm{pt}}(T_X) \neq \emptyset \iff L^{2,\infty} \subseteq X$.

(b) *The residual spectrum:* $\sigma_{\mathrm{r}}(T_X) \neq \emptyset \iff X \subseteq L^{2,1}$.

(c) *The continuous spectrum:*
$\sigma_{\mathrm{c}}(T_X) = \sigma(T_X) \iff L^{2,\infty} \nsubseteq X, X \nsubseteq L^{2,1}$.

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A function space where the norm (i.e., membership) of a function does not depend on its shape, but on its size (i.e., on the measure of its level sets).

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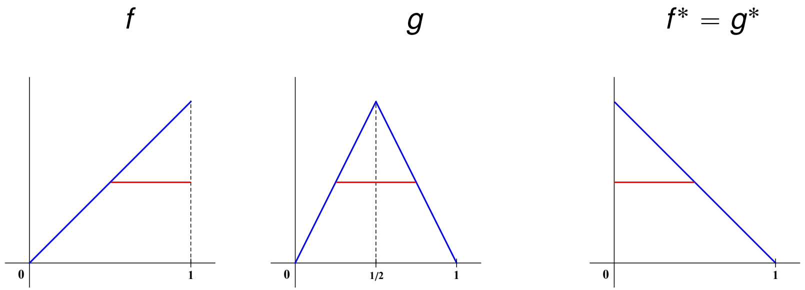

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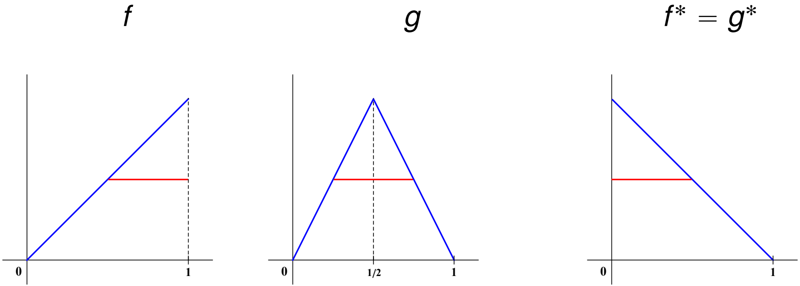

L^p spaces, weak L^p spaces, Orlicz spaces L^Φ, Lorentz $L^{p,q}$ spaces, Lorentz Λ_ϕ spaces, Marcinkiewicz $M(\varphi)$ spaces,.....

Boundedness of the FHT on r.i. spaces

- Theorem (Boyd): For H the Hilbert transform and X a r.i. space in $\mathbb{R}$:

$$H\colon X \to X \iff 0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$$

  where $\underline{\alpha}_X, \overline{\alpha}_X \in [0, 1]$ are the **Boyd indices of X**, which measure the effect of dilations on X:

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- Boyd's result extends to the FHT: for X a rearrangement invariant space in $(-1,1)$

$$T\colon X \to X \iff 0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1.$$

Outline

Inversion of the FHT on L^p

Let $w(x) := \sqrt{1 - x^2}$. This inversion result is known since long.

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Theorem

(i) *Let $1 < p < 2$. Given $g \in L^p$, a function $f \in L^p$ is a solution of the airfoil equation $T(f) = g$ if and only if*

$$f = -\frac{1}{w}\, T(gw) + \frac{C}{w}, \quad C \in \mathbb{C}.$$

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(ii) *Let $2 < p < \infty$. Let $g \in L^p$ satisfy $\int_{-1}^{1} \frac{g(x)}{w(x)}\,dx = 0$. Then, the airfoil equation $T(f) = g$ admits a unique solution $f \in L^p$ given by*

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Note: the above result excludes L^2.

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(ii) *Let $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1/2$. Let $g \in X$ satisfy $\int_{-1}^{1} \frac{g(x)}{w(x)}\, dx = 0$. Then, the airfoil equation $T(f) = g$ admits a unique solution $f \in X$ given by*

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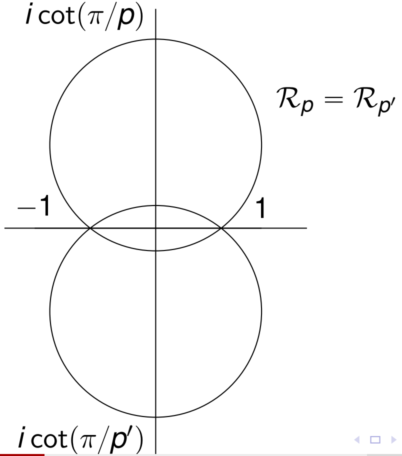
Note: the conditions above exclude spaces like L^2, $L^{2,1}$, $L^{2,\infty}$.

Outline

The spectrum of the FHT in L^p

$$\mathcal{R}_p := \{\pm 1\} \cup \left\{ \lambda \in \mathbb{C} : \frac{1}{2\pi} \left| \arg\left(\frac{1+\lambda}{1-\lambda} \right) \right| \leqslant \left| \frac{1}{2} - \frac{1}{p} \right| \right\}.$$



The spectrum of the FHT in L^p

Theorem (Widom, 1960)

Let $1 < p < \infty$. For the operator $T_p\colon L^p(-1,1) \to L^p(-1,1)$ we have

$$\sigma(T_p) = \mathcal{R}_p.$$

Moreover:

	$\sigma_{\mathrm{pt}}(T_p)$	$\sigma_{\mathrm{r}}(T_p)$	$\sigma_{\mathrm{c}}(T_p)$
$1 < p < 2$	$\mathrm{int}(\mathcal{R}_p)$	$\varnothing$	$\partial\mathcal{R}_p$
$p = 2$	$\varnothing$	$\varnothing$	$\mathcal{R}_2 = [-1,1]$
$2 < p < \infty$	$\varnothing$	$\mathrm{int}(\mathcal{R}_p)$	$\partial\mathcal{R}_p$

The point spectrum of T_X

Theorem (COR, 2021)

For a r.i. space X with $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$ we have

$$T_X \colon X \to X.$$

Let

$$p_X := \inf \left\{ p \in (1, \infty) : |x|^{-1/p} \in X \right\}.$$

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Then

(a) *If $p_X > 2$ or $p_X = 2$ and it is attained, then $\sigma_{\mathrm{pt}}(T_X) = \emptyset$.*

(b) *If $p_X \leqslant 2$ and p_X it is attained, then $\sigma_{\mathrm{pt}}(T_X) = \mathcal{R}_{p_X} \setminus \{\pm 1\}$.*

(c) *If $p_X < 2$ and it is not attained, then $\sigma_{\mathrm{pt}}(T_X) = \mathrm{int}(\mathcal{R}_{p_X})$.*

The residual spectrum of T_X

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For a r.i. separable space X with $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$ we have

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Let

$$q_X := \sup \left\{ q \in (1, \infty) : X \subseteq L^{q,1} \right\}.$$

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Then

(a) *If $q_X > 2$ and q_X is not attained, then $\sigma_{\mathrm{r}}(T_X) = \mathrm{int}(\mathcal{R}_{q_X})$.*

(b) *If $q_X \geqslant 2$ and q_X is attained, then $\sigma_{\mathrm{r}}(T_X) = \mathcal{R}_{q_X} \setminus \{\pm 1\}$.*

(c) *If $q_X < 2$ or $q_X = 2$ and it is not attained, then $\sigma_{\mathrm{r}}(T_X) = \varnothing$.*

The continuous spectrum of T_X

- The identification of the continuous spectrum of T_X is, in general, rather complicated.

The continuous spectrum of T_X

- The identification of the continuous spectrum of T_X is, in general, rather complicated.
- A partial result:

Theorem (COR, 2021)

Let X be a separable r.i.s. with $0 < \underline{\alpha}_X \leqslant \overline{\alpha}_X < 1$. Then

$$\sigma(T_X) \subseteq \mathcal{R}_{1/\underline{\alpha}_X} \cup \mathcal{R}_{1/\overline{\alpha}_X}.$$

The proof uses results on interpolation of operators.

The fine spectra of T_X

Theorem (COR, 2021)

Let X be a separable r.i. space with $0 < \underline{\alpha}_X = \overline{\alpha}_X < 1$. *Then*

$$\sigma(T_X) = \mathcal{R}_{p_X},$$

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	a.	$\mathcal{R}_{p_X}\setminus\{\pm 1\}$	$\varnothing$	$\{\pm 1\}$
$p_X > 2$	n.a.	$\varnothing$	$\operatorname{int}(\mathcal{R}_{p_X})$	$\partial\mathcal{R}_{p_X}$
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	q_X a.	$\varnothing$	$(-1, 1)$	$\{\pm 1\}$
	p_X, q_X n.a.	$\varnothing$	$\varnothing$	$\mathcal{R}_2 = [-1, 1]$

For the Lorentz $L^{p,r}$ spaces

Theorem (COR, 2021)

Let $1 < p < \infty$ *and* $1 \leqslant r < \infty$*. Then* $T_{p,r}\colon L^{p,r} \to L^{p,r}$ *and*

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Moreover:

$L^{p,r}$	$\sigma(T_{p,r}) = \mathcal{R}_p$	$\sigma_{\mathrm{pt}}(T_{p,r})$	$\sigma_{\mathrm{r}}(T_{p,r})$	$\sigma_{\mathrm{c}}(T_{p,r})$
$1 < p < 2$	$1 \leqslant r < \infty$	$\mathrm{int}(\mathcal{R}_p)$	$\varnothing$	$\partial\mathcal{R}_p$
$2 < p < \infty$	$r = 1$	$\varnothing$	$\mathcal{R}_p \setminus \{\pm 1\}$	$\{\pm 1\}$
	$1 < r < \infty$	$\varnothing$	$\mathrm{int}(\mathcal{R}_p)$	$\partial\mathcal{R}_p$
$p = 2$	$r = 1$	$\varnothing$	$(-1, 1)$	$\{\pm 1\}$
	$1 < r < \infty$	$\varnothing$	$\varnothing$	$[-1, 1]$

Outline

The Zygmund space $L\log L$

- The Zygmund space is

$$L\log L := \left\{ f\colon (-1,1) \to \mathbb{C} : \int_{-1}^{1} |f(x)| \log^{+} |f(x)|\, dx < \infty \right\}.$$

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- The finite Hilbert transform

$$T\colon L\log L \to L^1 \quad \text{is bounded.}$$

The Poincaré-Bertrand formula

Changing the order on integration in repeated Cauchy principal value integrals:

$$T\Big(gT(f) + fT(g)\Big) = T(f)T(g) - fg, \quad \text{a.e. on } (-1,1).$$

Hardy 1909, Poincaré 1909, Bertrand 1923, Muskhelishvili 1946, Tricomi 1957: $f \in L^p$, $g \in L^q$ with $1/p + 1/q < 1$, Love 1977: with $1/p + 1/q = 1$.

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Theorem (COR, 2024)

Let $f \in L^\infty$ and $g \in L\log L$. Then

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The pseudo-inverse $\widehat{T}$

- The pseudo-inverse for $T\colon L^p \to L^p$ when $1 < p < 2$ is

$$\widehat{T}g(x) := \frac{-1}{w(x)} T(gw)(x), \quad x \in (-1,1).$$

In 1958 Khvedelidze proved, for $1 < p < 2$, that

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- In order to apply **extrapolation** we need a precise bound on the norm

$$\|\widehat{T}\|_{L^p \to L^p} \quad \text{when } p \text{ is near } 1.$$

Bound on $\|T + \widehat{T}\|_{L^p \to L^p}$

Given $\beta, \gamma \in (0,1)$ with $1 < (\beta+\gamma)$ let

$$c(\beta,\gamma) := \max\left\{\frac{1}{1-\beta}, \frac{1}{1-\gamma}, \frac{1}{\beta+\gamma-1}\right\}.$$

Then we have:

- $\displaystyle\int_{-1}^{\infty} \frac{d\xi}{|\xi|^{\beta}(\xi+1)^{\gamma}} \leqslant 6\, c(\beta,\gamma)$
- $\displaystyle\int_{-1}^{1} \frac{dt}{|t-x|^{\beta}|1-t^2|^{\gamma}} \leqslant \frac{24\, c(\beta,\gamma)}{(1-x^2)^{\beta+\gamma-1}}, \quad x \in (-1,1).$
- If $1 < p < 2$ and δ satisfies $\frac{(p-1)}{2} < \delta p < \min\left\{\frac{1}{2}, (p-1)\right\}$, then

$$\|T + \widehat{T}\|_{L^p \to L^p} \leqslant \frac{24\sqrt{2}}{\pi}\left(c\left(\frac{1}{2}, \frac{1}{2} + \delta p\right)\right)^{1/p}\left(c\left(\frac{1}{2}, \delta p'\right)\right)^{1/p'}.$$

$\widehat{T}$ on $L\log L$

Theorem (COR, 2024) For $1 < p < 3/2$ we have

$$\|\widehat{T}\|_{L^p \to L^p} \leqslant \frac{72\sqrt{2}}{p-1} + \frac{3}{p-1}.$$

This result, together with a general version of **Yano's extrapolation theorem**, allows to prove

Theorem (COR, 2024) For $\beta \geqslant 0$, we have

$$\widehat{T} \colon L(\log L)^{1+\beta} \to L(\log L)^{\beta}$$

boundedly. In particular,

$$\widehat{T} \colon L\log L \to L^1 \quad \text{is bounded.}$$

T and $\widehat{T}$

The following diagram illustrates the action of T and $\widehat{T}$ in various spaces.

$$
\begin{tikzcd}
L(\log L)^3 \arrow[r, hook] \arrow[dr, "T"] & L(\log L)^2 \arrow[r, hook] \arrow[dr, "T"] & L(\log L) \arrow[dr, "T"] & \\
 & L(\log L)^2 \arrow[r, hook] & L(\log L) \arrow[r, hook] & L^1 \\
L(\log L)^3 \arrow[r, hook] \arrow[ur, "\widehat{T}"] & L(\log L)^2 \arrow[r, hook] \arrow[ur, "\widehat{T}"] & L(\log L) \arrow[ur, "\widehat{T}"] &
\end{tikzcd}
$$

Inversion of the FHT on $L\log L$

Theorem (COR, 2024)

(i) *The range of* $T\colon L\log L \to L^1$ *is*

$$T(L\log L) = \Big\{ g \in L^1 : \widehat{T}(g) \in L\log L,\ T(\widehat{T}(g)) = g \Big\}.$$

(ii) *For* $g \in T(L\log L)$ *all solutions* f *to the airfoil equation* $T(f) = g$ *are of the form*

$$f = \frac{-1}{w}\, T(wg) + \frac{C}{w}, \quad C \in \mathbb{C}.$$

(iii) *In particular, the above holds for* $g \in L(\log L)^2$.

Some refs: Curbera, Okada, Ricker

- *The finite Hilbert transform acting in the Zygmund space Llog L*, Annali della Scuola Normale Superiore di Pisa, 2024.
- *The fine spectra of the finite Hilbert transform in function spaces*, Advances in Mathematics, 2021.
- *Inversion and extension of the finite Hilbert transform on (-1,1)*, Annali di Matematica Pura ed Applicata, 2019.

Thank you for your attention

y... moltes felicitats, Pepe!