Projection constants of Banach spaces of multivariate polynomials on domains with generic group structures

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A Conference to honour the 70th Birthday of José Bonet

41 years ago

24 years later

Today

Back to mathematics ...

Meeting the team ...

Projection constants: The definition

Relative projection constant

Let X be a subspace of a Banach space Y. Then

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Absolute projection constant

$$\boldsymbol{\lambda}(X) := \sup\ \boldsymbol{\lambda}(X,Y)\,,$$

the supremum taken over all Y containing X as an isometric copy

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For every **finite-dimensional** subspace S of $C(K)$ we have

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and then there is a so-called **minimal projection** $P \in \mathcal{L}(C(K))$ onto S, that is,

$$\boldsymbol{\lambda}(S) = \|P : C(K) \to S\|$$

A first orientation:
The theorem of Kadets-Snobar

Kadets-Snobar Theorem 1971

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Hahn-Banach theorem

$$\boldsymbol{\lambda}(\ell_\infty^n) = 1$$

Aims of our project

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and more specifically,

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Our program includes applications to spaces of

- polynomials on fin.-dim. Hilbert spaces,
- trigonometric polynomials,
- **functions on Boolean cubes,**
- **Dirichlet polynomials,**
- **polynomials on $\mathcal{L}(\ell_2^n)$, ...**

Inspiring examples

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$$\boldsymbol{\lambda}\big(\mathbf{Trig}_{\leq d}(\mathbb{T})\big) = \frac{4}{\pi^2}\log(d+1) + o(1)$$

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- Ryll-Wojtaszczyk 1983: In the complex case

$$\boldsymbol{\lambda}\big(\mathcal{P}_d(\ell_2^n)\big) \;=\; \frac{\Gamma(n+d)\Gamma(1+\frac{d}{2})}{\Gamma(1+d)\Gamma(n+\frac{d}{2})}$$

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- D-Frerick 2011: For $1 \le r \le \infty$ and in the real/complex case

$$\boldsymbol{\lambda}\big(\mathcal{P}_d(\ell_r^n)\big) \;\sim_{C^d}\; \Big(1+\frac{n}{d}\Big)^{d\big(1-\frac{1}{\min\{r,2\}}\big)}$$

A conceptual scenario: one possibility

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- **Abelian case**: $\Phi_g : C(G) \to C(G)\,, \quad f(x) \mapsto f(g^{-1}x)$
- **Non-Abelian case**: $\Phi_{g,h} : C(G) \to C(G)\,, \quad f(x) \mapsto f(g^{-1}xh)$

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An extension of this setting to so-called homogeneous compact Hausdorff spaces is possible ...

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Theorem - team

S is G-invariant ⇔ S is G-accessible

Averaging projections

Theorem - team

For every fin.-dim. G-invariant subspace S of $C(G)$

$$\boldsymbol{\lambda}(S) = \left\|\boldsymbol{\pi}_S : C(G) \to S\right\|$$

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For every fin.-dim. G-invariant subspace S of $C(G)$

$$\boldsymbol{\lambda}(S) = \left\|\boldsymbol{\pi}_S : C(G) \to S\right\| = \int_G |\mathbf{k}_S(e, \cdot)| dm$$

Reproducing kernel

There is a unique continuous function $\mathbf{k}_S : G \times G \to \mathbb{C}$ such that for all $x \in G$

$$(\boldsymbol{\pi}_S f)(x) = \left\langle f, \mathbf{k}_S(x, \cdot)\right\rangle_{L_2(G)} \quad \text{for all } f \in L_2(G)$$

Proof of the first part

Given any projection $\mathbf{Q} : C(G) \to S$, we have to show that

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- Prove that

$$\boldsymbol{\pi}_S = \int_G \int_G \boldsymbol{\Phi}_{g,h}^{-1} \circ \mathbf{Q} \circ \boldsymbol{\Phi}_{g,h} \, dg \, dh : \; C(G) \to S$$

- Then the conclusion follows by the 'triangle inequality'.

Two Abelian examples

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Problem 1

$$\lambda(\mathcal{B}^N_{\leq d}) = ?$$

Theorem - team

$$\lambda\left(\mathcal{B}^N_{\leq d}\right) = \frac{1}{2^N} \sum_{x \in \{\pm 1\}^N} \Big| \sum_{\substack{A \subset \{1,\dots,N\} \\ |A| \leq d}} \prod_{n \in A} x_n \Big|$$

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and, fixing d,

$$\lim_{N \to \infty} \frac{\boldsymbol{\lambda}\big(\mathcal{B}^N_{\leq d}\big)}{\sqrt{\dim \mathcal{B}^N_{\leq d}}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{|h_d(t)|}{\sqrt{d!}} e^{-\frac{t^2}{2}}\, dt$$

$$= \frac{2^{7/4}}{\pi^{5/4}} \frac{1}{d^{1/4}} \left(1 + O\Big(\frac{1}{d^2}\Big)\right),$$

where h_d is the d-th Hermite polynomial

Problem 2

Find the projection constant

$$\boldsymbol{\lambda(\mathcal{H}_{\infty}^{\leq x}) = ?}$$

where $\boldsymbol{\mathcal{H}_{\infty}^{\leq x}}$ stands for the Banach space of all Dirichlet polynomials of length x:

$$\sum_{n \leq x} a_n n^{-s}, \quad s \in \mathbb{C}$$

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The asymptotic order of the integral is a recent deep result of Harper from 2019 in probabilistic analytic number theory ...

Two non-Abelian examples

Problem 3

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$$\boldsymbol{\lambda}\big(\mathcal{S}_1(\ell_2^n)\big) = \textbf{?}$$

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$$\frac{n}{3} \leq \boldsymbol{\lambda}\big(\mathcal{S}_1(\ell_2^n)\big) \leq n$$

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How to use the general scheme?

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$$\mathcal{S}_1(\ell_2^n) = \mathcal{L}(\ell_2^n)^* \hookrightarrow C(\mathcal{U}_n),\ \boldsymbol{V \mapsto [U \mapsto \mathrm{tr}(VU)]} \quad \text{isometrically}$$

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We so far don't know the analog result for $d = 3, 4, 5, \ldots$

Team work:

- Projection constants for spaces of Dirichlet polynomials; Mathematische Annalen 2024
- Asymptotic insights for projection, Gordon-Lewis, and Sidon constants of Boolean Cube function spaces; International Mathematical Research Notices 2024
- An integral formula for the projection constant of the trace class; Analysis&PDE 2024
- Minimal projections onto spaces of polynomials on real euclidean spheres; submitted 2025
- Projection constants of spaces of bihomogeneous polynomials on complex euclidean spheres; submitted 2025
- Local constants and Bohr's phenomenon for Banach spaces of analytic polynomials; submitted 2025
- Projection constants of spaces of multivariate polynomials; an arXiv manuscript 2022 that is gradually improved ...