Commutativity in non-elliptic iteration

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(based on joint works with M.D. Contreras [Universidad de Sevilla]
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- Centralizer of a function $\varphi \in \mathsf{Hol}(\mathbb{D})$:

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- Direct analysis: description of the centralizer [in general, completely unfeasible and usually not very useful].
- Indirect analysis: consider a qualitative approach to the centralizers.

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- This approach has a long story. At least, we must cite: Heins (1941), Pranger (1970), Shields (1964), Behan (1973), Cowen (1984), Gentili-Vlacci (1994), Bisi-Gentili (2001).

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- This approach has a long story. At least, we must cite: Heins (1941), Pranger (1970), Shields (1964), Behan (1973), Cowen (1984), Gentili-Vlacci (1994), Bisi-Gentili (2001).
- Cowen's paper has been fundamental in the current development of the theory.

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- In what follows, we assume that $\varphi \in \mathrm{Hol}(\mathbb{D})$ has no fixed point, that is, $\varphi(z) \neq z$, for all $z \in \mathbb{D}$. In other words, we assume that φ is non-elliptic.

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- In what follows, we assume that $\varphi \in \mathrm{Hol}(\mathbb{D})$ has no fixed point, that is, $\varphi(z) \neq z$, for all $z \in \mathbb{D}$. In other words, we assume that φ is non-elliptic.
- If $\varphi \in \mathrm{Hol}(\mathbb{D})$ is non-elliptic, there exists $\tau \in \partial\mathbb{D}$ such that, for every $z \in \mathbb{D}$, $\lim_{n\to\infty} \varphi_n(z) = \tau$. This point τ is (clearly) unique and it is called the Denjoy-Wolff point of φ.

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  and h is univalent in a certain "nice domain close to τ".
- It is always possible to find such a Koenigs function and, indeed, with the following property:

$$\Omega := \bigcup_{n=0}^{\infty} (h(\mathbb{D}) - n)$$

  is either a horizontal strip, a horizontal half-plane or $\mathbb{C}$. This subset Ω is called the base domain and the triple $(\Omega, h, z \mapsto z + 1)$ a canonical model of φ.

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- Canonical models always exist and they are essentially unique.

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- Dynamically speaking, a canonical model tell us that the iterates system

$$n \in \mathbb{N} \mapsto \varphi_n(z),\ z \in \mathbb{D} \quad // \quad n \in \mathbb{N} \mapsto w + n,\ w \in h(\mathbb{D})$$

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are equivalent.

- In other words, the dynamical information about (φ_n) is basically encoded in the geometry of $h(\mathbb{D})$.

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  - Clearly, for every $z \in \mathbb{D}$,

$$\psi \circ \varphi(z) = h_\varphi^{-1}(h_\varphi(z) + 1 + c) = h_\varphi^{-1}(h_\varphi(z) + c + 1) = \varphi \circ \psi(z),$$

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  - Note $h_\varphi \circ \psi = h_\varphi + c$ and $h_\varphi \circ \varphi = h_\varphi + 1$.

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    thus $\psi \in \mathcal{Z}(\varphi)$.
  - Note $h_\varphi \circ \psi = h_\varphi + c$ and $h_\varphi \circ \varphi = h_\varphi + 1$.
- Central question: until what point this “simple geometrical” procedure is the “unique” way to generate elements of $\mathcal{Z}(\varphi)$?

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- Parabolic positive: the base domain is the upper half-plane $\mathbb{H}$ (positive$^+$) or the lower half-plane $-\mathbb{H}$ (positive$^-$).
- Parabolic zero: the base domain is $\mathbb{C}$.

The hyperbolic case

Hyperbolic case I

Hyperbolic case I

Theorem (Heins)
Let $\varphi \in \mathsf{Hol}(\mathbb{D})$ *be a hyperbolic automorphism and let* h_φ *be the Koenigs function of* φ. *A function* $\psi \in \mathsf{Hol}(\mathbb{D})$ *different from* $\mathrm{id}_{\mathbb{D}}$ *commutes with* φ *if and only if there exists a number* $c \in \mathbb{R}$ *such that* $h_\varphi \circ \psi = h_\varphi + c$.

Hyperbolic case II

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Theorem (Cowen;Behan,Gentili,Vlacci,Bisi)
Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ be hyperbolic, different from an automorphism, with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ and let h_φ be the Koenigs function of φ. A function $\psi \in \mathrm{Hol}(\mathbb{D})$ different from $\mathrm{id}_\mathbb{D}$ commutes with φ if and only if the following two conditions hold:

1. *The Denjoy-Wolff point of ψ is τ.*
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1. *The Denjoy-Wolff point of ψ is τ.*
2. *There exists a number $c \in \mathbb{R}$ such that $h_\varphi \circ \psi = h_\varphi + c$.*

Moreover, if h_φ is univalent, then a function $\psi \in \mathrm{Hol}(\mathbb{D})$ different from $\mathrm{id}_\mathbb{D}$ commutes with φ if and only there exists a number $c \in \mathbb{R}$ such that $h_\varphi \circ \psi = h_\varphi + c$.

Hyperbolic case III

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- The previous number c is clearly unique and it is called the constant of simultaneous linearisation of ψ with respect to φ and it is denoted as $c_{\varphi,\psi}$.

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- Indeed, if $\psi \in \mathcal{Z}(\varphi)$,

$$c_{\varphi,\psi} = \frac{\log(\psi'(\tau))}{\log(\varphi'(\tau))}.$$

The parabolic zero case

Parabolic zero case I

Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathsf{Hol}(\mathbb{D})$ *be parabolic zero with Denjoy-Wolff point* $\tau \in \partial\mathbb{D}$ *and let* h_φ *be the Koenigs function of* φ*. A function* $\psi \in \mathsf{Hol}(\mathbb{D})$ *different from* $\mathrm{id}_\mathbb{D}$ *commutes with* φ *if and only if if the following two conditions hold:*

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- There is no a simple formula for $c_{\varphi,\psi}$ as in the hyperbolic case.

The parabolic positive case

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Theorem (Heins)
Let $\varphi \in \mathrm{Hol}(\mathbb{H})$ *be a parabolic automorphism (positive*$^+$*) and let* h_φ *be the Koenigs function of* φ*. A function* $\psi \in \mathrm{Hol}(\mathbb{H})$ *commutes with* φ *if and only if one of the mutually disjoint situations happens:*

1. *There exists a number* $c \in \mathbb{R}$ *such that*

$$\psi = h_\varphi^{-1} \circ (h_\varphi + c).$$

2. *There exists* $F \in \mathrm{Hol}(\mathbb{D}, \mathbb{H})$ *such that*

$$\psi(w) = h_\varphi^{-1} \circ (h_\varphi(w) + F(e^{2\pi i h_\varphi(w)})), \ w \in \mathbb{H}.$$

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- Our central question has a negative answer in this context.

Parabolic positive case II

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Parabolic positive case II

- Central question reformulated: In the parabolic positive case, is there any kind of simultaneously linearization result for two commuting elements?
- For the sake of clarity, we will restrict to the case of parabolic positive$^+$ thus with a canonical model with base domain $\mathbb{H}$.

Parabolic positive case III

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Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ *be parabolic positive*$^{+}$ *with Koenigs function* h_φ *and let* $\psi \in \mathcal{Z}(\varphi)$*. Then, there exist* $\beta \in \mathrm{Hol}(\mathbb{H}, \mathbb{C})$ *and a number* $c \in \mathbb{C}$ *such that the following three conditions hold:*

1. $\beta(w+1) = \beta(w) + 1$, *for all* $w \in \mathbb{H}$.
2. $\beta \circ h_\varphi \circ \varphi = \beta \circ h_\varphi + 1$.
3. $\beta \circ h_\varphi \circ \psi = \beta \circ h_\varphi + c$.

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1. *If* $(\beta_1, c_1), (\beta_2, c_2) \in \mathrm{Hol}(\mathbb{H}, \mathbb{C}) \times \mathbb{C}$ *are two pairs satisfying the previous three conditions, then* $c_1 = c_2 \in \overline{\mathbb{H}}$*. This allows to define such a unique constant as the constant of simultaneous linearisation of* ψ *with respect to* φ *and it is denoted as* $c_{\varphi,\psi}$*.*

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2. *If* $(\beta_1, c), (\beta_2, c) \in \mathrm{Hol}(\mathbb{H}, \mathbb{C}) \times \mathbb{C}$ *are two pairs satisfying the previous three conditions with* $c \notin \mathbb{Q}$*, then there exists* $d \in \mathbb{C}$ *such that*
$$\beta_1 = \beta_2 + d.$$

Parabolic positive case V

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Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathsf{Hol}(\mathbb{D})$ *be parabolic positive*$^+$ *with Denjoy-Wolff point* $\tau \in \partial\mathbb{D}$*.*
A function $\psi \in \mathsf{Hol}(\mathbb{D})$ *different from* $\mathrm{id}_{\mathbb{D}}$ *commutes with* φ *if and only if the following two conditions hold:*

1. *The Denjoy-Wolff point of* ψ *is* τ*.*
2. *There exists a number* $c \in \overline{\mathbb{H}}$ *and* $h = h(\psi) \in \mathsf{Hol}(\mathbb{D}, \mathbb{C})$ *univalent in some truncated Stolz angular region of vertex* τ *such that* $h \circ \psi = h + c$ *and* $h \circ \varphi = h + 1$*.*

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- In general, h is really different from h_φ. Indeed, either $h = h_\varphi + d$, for some $d \in \mathbb{C}$ or $h \neq T \circ h_\varphi$ for any lineal fractional map T.

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- There is no a simple formula for $c_{\varphi,\psi}$ as in the hyperbolic case.

The parabolic case: constant of simultaneous linearisation

Constant of simultaneous linearisation I

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Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ *be parabolic and let* $\psi \in \mathcal{Z}(\varphi)$.

1. $c_{\varphi,\psi} = 0$ *if and only if* $\psi = id_{\mathbb{D}}$.
2. *If* $c_{\varphi,\psi} = \frac{m}{n} \in$ *with* $m, n \in \mathbb{N}$, *then* $\psi^{o(n)} = \varphi^{o(m)}$.
3. *If* $c_{\varphi,\psi} \in (-\infty, 0)$, *then* φ *as well as* ψ *are parabolic automorphisms. Moreover, if* $c_{\varphi,\psi} = -\frac{m}{n} \in \mathbb{Q}$ *with* $m, n \in \mathbb{N}$, *then* $\psi^{o(n)} = \varphi^{o(-m)}$.

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- From above, we can think of any $\psi \in \mathcal{Z}(\varphi)$ as a $c_{\varphi,\psi}$-iterate of φ.

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Constant of simultaneous linearisation II

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Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ *be parabolic.*

1. *Assume* $\psi_1, \psi_2 \in \mathcal{Z}(\varphi)$. *Then*

$$c_{\varphi,\psi_1\circ\psi_2} = c_{\varphi,\psi_1} + c_{\varphi,\psi_2} = c_{\varphi,\psi_2\circ\psi_1}.$$

   [in general, $\psi_1 \circ \psi_2 \neq \psi_2 \circ \psi_1$*].*

2. *Assume* $\psi \in \mathcal{Z}(\varphi)$ *is different from* $\mathrm{id}_{\mathbb{D}}$. *Then*

$$c_{\psi,\varphi} c_{\varphi,\psi} = 1.$$

Constant of simultaneous linearisation III

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Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ *be parabolic with Koenigs function* h_φ *and Denjoy-Wolff point* $\tau \in \partial\mathbb{D}$*. Let* $\psi \in \mathcal{Z}(\varphi)$*. Then,*

$$c_{\varphi,\psi} = \angle \lim_{z\to\tau} (h_\varphi \circ \psi(z) - h_\varphi(z)) = \angle \lim_{z\to\tau} h'_\varphi(z)(\psi(z) - z).$$

Constant of simultaneous linearisation IV

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Theorem (Contreras,Gumenyuk,DM)
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Constant of simultaneous linearisation V

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Theorem (Contreras,Gumenyuk,DM)
Let $\varphi \in \mathrm{Hol}(\mathbb{D})$ *be non-elliptic with Denjoy-Wolff point* $\tau \in \partial\mathbb{D}$ *and let* $\psi \in \mathcal{Z}(\varphi)$*. Then*

$$c_{\varphi,\psi} = \lim_{r \to 1^-} \frac{\log(\psi'(r\tau))}{\log(\varphi'(r\tau))}.$$

Abelian character of the centralizer

Abelian character I

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If φ is hyperbolic, then $[\mathcal{Z}(\varphi), \circ]$ is abelian.

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If φ is parabolic of zero hyperbolic step, then $[\mathcal{Z}(\varphi), \circ]$ is abelian.

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- It was known from the work of Heins that, in general, $[\mathcal{Z}(\varphi), \circ]$ is not abelian, when φ is parabolic of positive hyperbolic step. However:

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- It was known from the work of Heins that, in general, $[\mathcal{Z}(\varphi), \circ]$ is not abelian, when φ is parabolic of positive hyperbolic step. However:

Theorem (Contreras,Gumenyuk,DM)
If φ is parabolic of positive hyperbolic step, $\psi \in \mathcal{Z}(\varphi)$ and $c_{\varphi,\psi} \notin \mathbb{R}$ then

$$\{\phi \in \mathcal{Z}(\varphi) : \psi \circ \phi = \phi \circ \psi\}$$

is an abelian subsemigroup of $\mathcal{Z}(\varphi)$.

THANK YOU FOR YOUR ATTENTION

¡¡ FELIZ CUMPLEAÑOS, PEPE !!