Ultradifferentiable regularity of CR mappings

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The Schwarz Reflection Principle

Theorem

Let h be a holomorphic function on the upper halfplane which extends continuously to a real-valued function on the real line. Then there is an entire function F such that $F = f$ on the closed upper plane.

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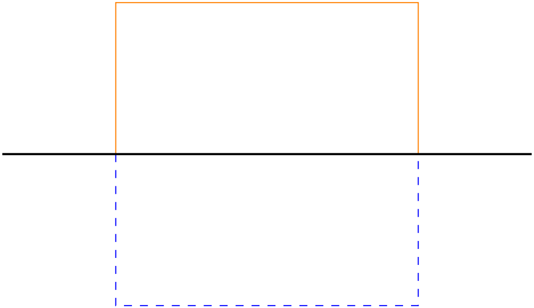

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  - There is a microlocal interpretation: If $u \in \mathcal{D}'(\mathbb{R})$ then by definition

    $$(x, -1) \notin \mathrm{WF}_A\, u \Longleftrightarrow (x, +1) \notin \mathrm{WF}_A\, \bar{u} \qquad \forall x \in \mathbb{R}.$$

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Theorem (Fefferman 1974)

Let $D \subseteq \mathbb{C}^N$ and $D' \subseteq \mathbb{C}^N$ be two bounded strongly pseudoconvex domains with smooth boundaries. Then every biholomorphic mapping between D and D' extends to a smooth diffeomorphism of the boundaries.

Introduction: Real Submanifolds of $\mathbb{C}^N$

Let J be the complex structure operator on $\mathbb{C}^N$:

$$J(Z) = \overline{Z} \qquad Z \in \mathbb{C}^N.$$

If M is a real submanifold of $\mathbb{C}^N$ with tangent space $T_pM \subseteq \mathbb{C}^N$ at $p \in M$ then

$$T_p^c M = T_pM \cap J(T_pM)$$

is the *complex* tangent space of M at p.

$$M \subseteq \mathbb{C}^N \text{ is CR } :\Longleftrightarrow \quad M \ni p \mapsto \dim_{\mathbb{C}} T_p^c M \text{ is constant.}$$

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Lemma

If $M \subseteq \mathbb{C}^{n+1}$ is a real hypersurface then M is a CR manifold of CR dimension n.

CR vector fields

If $\mathbb{C}T_p\mathbb{C}^N = \mathbb{C} \otimes T_p\mathbb{C}^N \cong \mathbb{C}^{2N}$ then

$$\frac{\partial}{\partial Z_j}\Big|_p = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\Big|_p - i\frac{\partial}{\partial y_j}\Big|_p\right), \quad \frac{\partial}{\partial \overline{Z}_j}\Big|_p = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\Big|_p + i\frac{\partial}{\partial y_j}\Big|_p\right),$$

where $j = 1, \ldots, N$, is a basis.
Let $T_p^{(0,1)} = \operatorname{span}\{\partial_{\overline{Z}_1}, \ldots, \partial_{\overline{Z}_N}\} \subseteq \mathbb{C}T_p\mathbb{C}^N$ and denote by $T^{(0,1)}$ the associated bundle.

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Definition

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is the CR bundle of M (note that $\mathrm{Re}\,\mathcal{V} = T^cM$).

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  is the CR bundle of M (note that $\operatorname{Re}\mathcal{V} = T^cM$).
- A CR vector field of M is a section L of $\mathcal{V}$ over M.
  We write $L \in \mathcal{V}$.

CR functions and CR mappings

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- A function (distribution) f on a CR manifold M is a CR function if $Lf = 0$ for all CR vector fields L of M.

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- If M is a real hypersurface and F a holomorphic function on one side of M which extends continuously to M then $F|_M$ is a CR function.
- If $h = (h_1, \dots, h_{N'}) : M \to M'$ is a CR mapping then h_j is a CR function on M for all $j = 1, \dots, N'$.

Remark

Let $h : M \to M'$ be a CR mapping with $n = \dim_{CR} M$ and $p \in M$. Suppose that there is a basis of CR vector fields $L_1, \ldots, L_n$ defined near p and let ρ' be a local defining function of M' near $h(p)$.

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Then the question of the regularity of the CR map h near p can be formulated as a regularity problem for the following overdetermined system of PDEs with nonlinear side conditions:

$$(\star) = \begin{cases} L_j h_k(q) & = 0, \qquad j = 1, \ldots, N,\ k = 1, \ldots, N', \\ \rho'(h(q)) & = 0 \qquad q \text{ near } p. \end{cases}$$

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Problem

If the data of $(\star)$ (i.e. $L_1, \ldots, L_n, \rho'$) is $\mathcal{C}^\infty$ ($\mathcal{C}^\omega$, etc.), under which conditions is the solution h of $(\star)$ $\mathcal{C}^\infty$ ($\mathcal{C}^\omega$, etc.)?

Further definitions

- Let $M \subseteq \mathbb{C}^{n+1}$ and $M' \subseteq \mathbb{C}^{n'+1}$ be real hypersurfaces and $h : M \to M'$ be a CR mapping defined near $p_0 \in M$.
- Assume that $L_1, \dots, L_n$ is a local basis of CR vector fields near a point p_0 and ρ' a local defining function of M' near $q_0 \in H(p_0)$.

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If $Z' = (Z_1, \dots, Z'_{n'+1})$ denote the coordinates of $\mathbb{C}^{n'+1}$ then we write $\partial\rho'/\partial Z' = (\partial\rho'/\partial Z'_1, \dots, \partial\rho'/\partial Z_{n'+1})$. If h is of class $\mathcal{C}^k$ then we set

$$E_k(p_0) = \operatorname{span}\left\{ L^\alpha \left(\frac{\partial\rho}{\partial Z'}(h(Z)) \right) \Bigg|_{Z=p_0} : \alpha \in \mathbb{N}_0^n,\ |\alpha| \le k \right\}.$$

Here $L^\alpha = L_1^{\alpha_1} \dots L_n^{\alpha_n}$.
Obviously, $E_k(p_0) \subseteq E_{k+1}(p_0)$ when h is of class $\mathcal{C}^{k+1}$.

Finitely nondegenerate CR mappings

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The CR mapping h is k_0-nondegenerate at p_0 if h is of class $\mathcal{C}^{k_0}$ near p_0 and $E_{k_0-1}(p_0) \subsetneq E_{k_0}(p_0) = \mathbb{C}^{n'+1}$.

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Theorem (Lamel 2001/04)

Let $M \subseteq \mathbb{C}^{n+1}$, $M' \subseteq \mathbb{C}^{n'+1}$ be real hypersurfaces and $k_0 \in \mathbb{N}$.
Suppose that $h : M \to M'$ is a CR mapping which is k_0-nondegenerate at $p_0 \in M$ and which extends continuously to a holomorphic mapping on one side of M.
If M and M' are smooth (real-analytic) then h is smooth (real-analytic) near p.

Main Ingrediants of the proof

- Using suitable charts of M and extends them in an almost-analytic way to a diffeomorphism.
- Fourier transforms.
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(e.g. Gevrey classes $\mathcal{G}^s, \ s \geq 1$.)
If M, M' are $\mathcal{A}$-manifolds, is H of class $\mathcal{A}$?
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- $\mathcal{A}$ has to be closed under composition and satisfying the implicit function theorem.
- $\mathcal{A}$ has to be microlocalizable, i.e. for a distribution u the $\mathcal{A}$-wavefront set $\mathrm{WF}_{\mathcal{A}}\, u$ can be defined.

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- $\mathcal{A}$ has to be microlocalizable, i.e. for a distribution u the $\mathcal{A}$-wavefront set $\mathrm{WF}_{\mathcal{A}}\, u$ can be defined.
- The algebra $\mathcal{A}$ can be characterized by almost-analytic extensions.

Classes given by weight matrices

- We say that a squence $\mathbf{M} = (M_k)_k$ of positive numbers is a weight sequence if $M_0 = 1$ and $M_k^2 \leq M_{k-1}M_{k+1}$ for all $k \in \mathbb{N}$.

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- A weight matrix $\mathfrak{M}$ is a family of weight matrices such that for any pair $\mathbf{M}, \mathbf{N} \in \mathfrak{M}$ we have either $M_k \leq N_k$, $\forall k$, or $M_k \leq N_k$, $\forall k$.

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Let $f \in \mathcal{C}^\infty(\Omega)$ and $\mathfrak{M}$ be a weight matrix.
Then $f \in \mathcal{E}^{\{\mathfrak{M}\}}(\Omega)$ iff

$$\forall\, K \Subset \Omega \quad \exists\, \mathbf{M} \in \mathfrak{M} \quad \exists C, h > 0 :$$
$$\sup_{x \in K} |D^\alpha f(x)| \leq C h^{|\alpha|} M_{|\alpha|} \qquad \forall\, \alpha \in \mathbb{N}_0^n.$$

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These classes include (Rainer–Schindl 2016)

- Denjoy-Carleman classes given by single weight sequences.
- Braun-Meise-Taylor classes given by weight functions.

Normal weight matrices

A weight matrix $\mathbf{M}$ is *normal* if the following conditions hold (where $m_k = M_k/k!$):

$$\forall\, \mathbf{M} \in \mathfrak{M} : \quad m_k^2 \le m_{k-1} m_{k+1} \quad \forall k \in \mathbb{N} \tag{1}$$

$$(m_k)^{1/k} \to \infty \qquad \text{for } k \to \infty, \tag{2}$$

$$\begin{gathered} \forall\, \mathbf{M} \in \mathfrak{M}\ \exists\, \mathbf{N} \in \mathfrak{M}\ \exists C > 0 : \\ M_{k+\ell} \le C^{k+\ell+1} N_k N_\ell \quad \forall k, \ell \in \mathbb{N}_0. \end{gathered} \tag{3}$$

Definition

Let $\mathbf{M}$ be a sequence satisfying (1). The weight associated to $\mathbf{M}$ is

$$h_{\mathbf{M}}(t) = \inf_k m_k t^k.$$

Remarks

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- Every function f of class $\{\mathfrak{M}\}$ can locally be $\{\mathfrak{M}\}$-almost analytically extended (F.–Nenning–Rainer–Schindl 2020):

  A distribution f is a function of class $\{\mathfrak{M}\}$ near $p_0 \in \mathbb{R}^n$ if and only if there are a nbhd U of p_0, a smooth function $F \in \mathcal{C}^\infty(U + i\mathbb{R}^n)$ with $F|_U = f|_U$ and $\mathbf{M} \in \mathfrak{M}$, $C, h > 0$ such that for all $x + iy \in U \times \mathbb{R}^n$:

$$\left|\bar{\partial} F(x+iy)\right| \leq C h_{\mathbf{M}}(Q|y|) \tag{4}$$

The ultradifferentiable wavefront set

We can use (4) to microlocalize:

Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$. $(x_0, \xi_0) \notin \mathrm{WF}_{\{\mathfrak{M}\}}\, u :\Longleftrightarrow$ There are a nbhd $U \subseteq \mathbb{R}^n$ of x_0, open convex cones $\Gamma^1, \dots, \Gamma^N$ with $\xi_0 \Gamma^j < 0$, $j = 1, \dots, N$, and functions $F^j \in \mathcal{C}^\infty(U + i\Gamma^j)$, $j = 1, \dots, N$, such that each F^j satisfies (4) in $U \times \Gamma^j$ and

$$u|_U = \sum_{j=1}^{N} \operatorname{bv} F^j.$$

In fact, we can define $\mathrm{WF}_{\{\mathfrak{M}\}}\, u \subseteq T^*M$ for distributions $u \in \mathcal{D}'(M)$ on manifolds M of class $\{\mathfrak{M}\}$ (F.–Nenning–Rainer–Schindl 2020).
In particular if $\Phi : M \to N$ then $\Phi^*(\mathrm{WF}_{\{\mathfrak{M}\}}\, u) = \mathrm{WF}_{\{\mathfrak{M}\}}(\Phi^* u)$ for all $u \in \mathcal{D}'(N)$.

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- If $u \in \mathcal{D}'(M)$ then $\pi_1(\mathrm{WF}_{\{\mathfrak{M}\}}\, u) = \operatorname{sing\,supp}_{\{\mathfrak{M}\}} u$.
- If $\mathfrak{M}$ is a normal weight matrix then the microlocal elliptic theorem holds for $\mathrm{WF}_{\{\mathfrak{M}\}}$: Suppose that $P(x,D)$ is linear differential operator with coefficients in $\mathcal{E}^{\{\mathfrak{M}\}}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$ Then

$$\mathrm{WF}_{\{\mathfrak{M}\}}\, Pu \subseteq \mathrm{WF}_{\{\mathfrak{M}\}}\, u \subseteq \mathrm{WF}_{\{\mathfrak{M}\}}\, Pu \cup \operatorname{Char} P.$$

(F.–Nenning–Rainer–Schindl 2020; for BMT-Classes cf. Albanese–Jornet–Oliaro 2010)

Remarks II

Almost analytic functions with parameters

Let $\mathfrak{M}$ be a regular weight matrix, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^d$ be open sets and $\Gamma \subseteq \mathbb{R}^d$ an open convex cone.
We suppose that $F \in \mathcal{C}^\infty(U \times V \times \Gamma)$ is $\{\mathfrak{M}\}$-almost analytic in $(s,t) \in V \times \Gamma$, i.e. there is some $\rho > 0$ such that for each compact $K \times L \subseteq U \times V$ there are $\mathbf{M} \in \mathfrak{M}$ and $C, Q > 0$ so that

$$\left|\bar{\partial}_w F(x,s,t)\right| \leq C h_{\mathbf{M}}(Q|t|), \qquad (x,s) \in K \times L,\ |t| < \delta$$

where $w = s + it \in \mathbb{C}^d$.

Then $u = \operatorname{bv} F := \lim_{t \to 0} F(\,.\,,\,.\,,t) \in \mathcal{D}'(U \times V)$ and

$$\operatorname{WF}_{\{\mathfrak{M}\}} u = (U \times V) \times \mathbb{R}^n \times \Gamma^\circ.$$

Here $\Gamma^\circ = \{\xi \in \mathbb{R}^d \ : \ \xi y \geq 0 \quad \forall\, y \in \Gamma\}$

Applications for CR functions

Let $M \subseteq \mathbb{C}^{n+1}$ be a real hypersurface with CR bundle $\mathcal{V}$ of class $\{\mathfrak{M}\}$, $\mathfrak{M}$ being normal.
The characteristic bundle of M is

$$T^0M = \{\omega \in T^*M \,:\, \xi(L) = 0 \;\; \forall\, L \in \mathcal{V}\}.$$

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Local coordinates

If $p_0 \in M \subseteq \mathbb{C}^{n+1}$ then there are holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ in a nbhd U of p_0, vanishing at p_0, such that

$$M \cap U = \{(z, w) \in U \;:\; \operatorname{Im} w = \varphi(z, \operatorname{Re} w)\}$$

where $\varphi \in \mathcal{E}^{\{\mathfrak{M}\}}$ is a real-valued function and $\varphi(0) = 0 = d\varphi(0)$.

Local setting

- $\Phi : (x, y, s) \mapsto (x + iy, s + i\varphi(z, s))$ is a local parametrisation of M near $p_0 = 0$.
- In the local coordinates (x, y, s) of M near $p_0 = 0$ we have that

$$L_j = \frac{\partial}{\partial \bar{z}_j} - i \frac{\partial_{\bar{z}_j} \varphi(x, y, s)}{1 + i\partial_s \varphi(x, y, s)} \frac{\partial}{\partial s}, \qquad j = 1, \dots, n, \tag{5}$$

  is a basis of the CR vector fields near p_0.
- Thus $T_0^0 M \cong \{(0, 0, \sigma) \in \mathbb{R}^{2n+1} \ : \ \sigma \in \mathbb{R}\}$.

The wavefront set of a CR function

- If u is a CR distribution defined near p_0 and $\tilde{u} = \Phi^*(u)$ we see that $\mathrm{WF}_{\{\mathfrak{M}\}}\, u|_{p_0} \cong \mathrm{WF}_{\{\mathfrak{M}\}}\, \Phi^*(u)|_0 \subseteq \left\{(0,0,\sigma) \in \mathbb{R}^{2n+1} \,:\, \sigma \in \mathbb{R}\backslash\{0\}\right\}$

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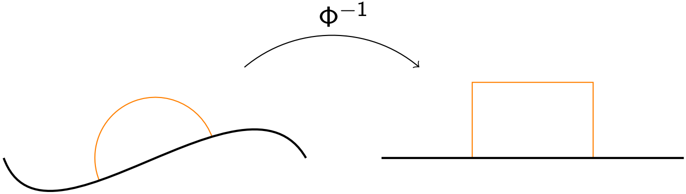

The wavefront set of a CR function

- $\tilde{H}$ is defined on a set of the form $U \times (0, \varepsilon)$, $U \subseteq \mathbb{R}^{2n+1}$ and $\varepsilon > 0$ and can continuously extended to $U \times \{0\}$ with $\tilde{H}|_{U\times\{0\}} = \tilde{u}$
- It follows directly that

$$\mathrm{WF}_{\{\mathfrak{M}\}}\, \tilde{u}|_0 \subseteq \mathbb{R}^{2n} \times \{\sigma \geq 0\}.$$

- Hence

$$\mathrm{WF}_{\{\mathfrak{M}\}}\, \tilde{u}|_0 = \big\{(0, 0, \sigma) \in \mathbb{R}^{2n+1} \,:\, \sigma > 0\big\}.$$

The Main Statement

Theorem (F. 2017/2020, F.–Lamel 2025)

Let $M \subseteq \mathbb{C}^{n+1}$ and $M' \subseteq \mathbb{C}^{n'+1}$ be two real hypersurfaces of class $\{\mathfrak{M}\}$, $p_0 \in M$ and $h : M \to M'$ be a $\mathcal{C}^{k_0}$-CR mapping that is k_0-nondegenerate at p_0.
Suppose furthermore that h extends continuously to a holomorphic map H on one side of M.
Then h is ultradifferentiable of class $\{\mathfrak{M}\}$ near p_0.

Sketch of proof

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- Then the vector fields $L_1, \dots, L_n$ given by (5) form a basis for the CR vector fields of M near p_0.
- Let ρ' a local defining function of M' near $h(0)$ then obviously $\rho' \circ h = 0$ near $0 \in \mathbb{R}^{2n+1}$.

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For $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k_0$, consider

$$L^\alpha \left(\rho' \circ h\right)(x,y,s) = \Psi_\alpha \left(h(x,y,s), \left(L^\beta \overline{h}(x,y,s) \right)_{|\beta| \leq k_0} \right) = 0 \qquad (6)$$

in the local coordinates $(x,y,s) \in \mathbb{R}^{2n+1}$. Here Ψ_α is a function of class $\{\mathfrak{M}\}$ defined in some open set in $\mathbb{C}^{n'+1} \times \mathbb{C}^K$ which is polynomial in the last variables.

Sketch of the proof

Moreover,

$$L^{\alpha}\left(\frac{\partial \rho'}{\partial Z'}\left(L^{\beta}\overline{h}(0,0,0)\right)_{|\beta|\le k_0}\right)=\frac{\partial \Psi_{\alpha}}{\partial Z'}\left(h(0,0,0),\left(L^{\beta}\overline{h}(0,0,0)\right)_{|\beta|\le k_0}\right)$$

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- By assumption there are multi-indices $\alpha^1,\ldots,\alpha^{n'+1}\in\mathbb{N}_0^n$ such that the matrix
$$\left(\frac{\partial\Psi_{\alpha^1}}{\partial Z'},\ldots,\frac{\partial\Psi_{\alpha^{n'+1}}}{\partial Z'}\right)$$
is invertible at $(h(0,0,0),(L^\beta\overline{h}(0,0,0))_{|\beta|\le k_0})$.
- Using the (smooth) implicit function theorem to "solve" the equation $\Psi(Z',\Lambda)=0$, $(\Psi=(\Psi_{\alpha^1},\ldots,\Psi_{\alpha^{n'+1}}))$ in a "particular" way we can show that there is a smooth mapping $\psi:\mathbb{C}^{n'+1}\times\mathbb{C}^K\to\mathbb{C}^{n'+1}$ defined near $(0,(L^\beta\overline{h}(0))_{|\beta|\le k_0})$ such that (by (6))
$$h(x,y,s)=\psi\left(h(x,y,s),\left(L^\beta\overline{h}(x,y,s)\right)_{|\beta|\le k_0}\right)$$
and the following holds:

Sketch of proof

First, ψ is holomorphic in the last K variables. Recall that by assumption of the theorem we have that

- there is a smooth extension H of h on $U \times I \times (0, \varepsilon)$, $U \subseteq \mathbb{R}^{2n}$, $I \subseteq \mathbb{R}$, which is $\{\mathfrak{M}\}$-almost analytic in the last variables $s + it$.
- Hence $\bar{h}$ extends to an $\{\mathfrak{M}\}$-almost analytic mapping $\tilde{h}$ on $U \times I \times (-\varepsilon, 0)$.
- Similarly, for each $\beta \in \mathbb{N}_0^n$, $L^\beta \bar{h}$ extends to a $\{\mathfrak{M}\}$-almost analytic mapping $\tilde{H}_\beta$ on $U \times I \times (-\varepsilon, 0)$.

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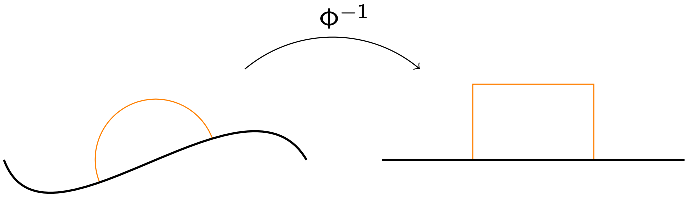
Then we can show that

$$G(x,y,s,t) = \psi\left(H(x,y,s,-t), \left(\tilde{H}_\beta(x,y,s,t)\right)_{|\beta|\leq k_0}\right)$$

is $\{\mathfrak{M}\}$-almost analytic on $U \times I \times (-\varepsilon, 0)$.
(We may have to shrink U, I and ε.)

Sketch of proof



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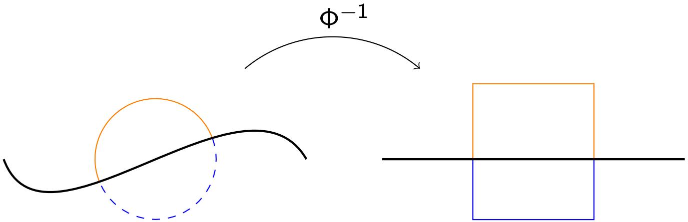

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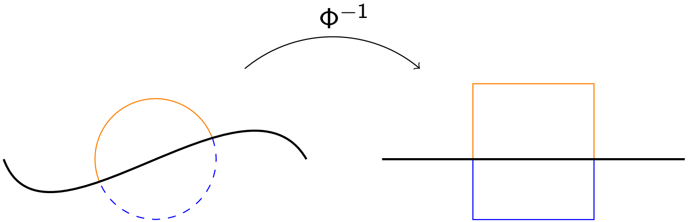

It follows that

$$\mathrm{WF}_{\{\mathfrak{M}\}} h|_0 = \bigcup_{j=1}^{n'+1} \mathrm{WF}_{\{\mathfrak{M}\}} h_j \subseteq (\{0\} \times \{\sigma > 0\}) \cap (\{0\} \times \{\sigma < 0\}) = \emptyset$$

$$\Longrightarrow \quad 0 \notin \operatorname{sing\,supp}_{\{\mathfrak{M}\}} h = \bigcup_{j=1}^{n'+1} \operatorname{sing\,supp}_{\{\mathfrak{M}\}} h_j.$$

□

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In order to present one of their theorems we need some definitions:

Definition

A CR mapping $H : M \to M'$ is *strictly noncharacteristic* if

$$H^* \left(T^0_{H(p)} M' \right) = T^0_p M \qquad \forall p \in M.$$

D'Angelo finite type

Let $M \subseteq \mathbb{C}^{n+1}$ be a real hypersurface, $p \in M$ and ρ a local defining function of M near p. We set

$$\Delta(M,p) := \sup_{\substack{\gamma:\mathbb{D}\to\mathbb{C}^{n+1} \\ \gamma(0)=p,\ \gamma\not\equiv p}} \frac{\nu_0(\rho\circ\gamma)}{\nu_0(\gamma)} \in \mathbb{R}\cup\{\infty\}.$$

Here $\mathbb{D} \subseteq \mathbb{C}$ is the unit disc, $\nu_0(\gamma)$ is the vanishing order of γ at 0.

Definition

We say that M is of finite type at p if $\Delta(M,p) < \infty$.
If $\Delta(M,p) = \infty$ then M is of infinite type at p.

$$\mathcal{I}_M := \{p \in M \ : \ \Delta(M,p) = \infty\} \subseteq M.$$

Geometric conditions for regularity

Theorem (Lamel–Mir 2018)

Let $M \subseteq \mathbb{C}^{n+1}$, $M' \subseteq \mathbb{C}^{n'+1}$ be real hypersurfaces with $n' > n \geq 1$. Suppose that:

- *M and M' are of class $\mathcal{C}^\infty$.*
- *M is strongly pseudoconvex.*
- *There is a strongly non-characteristic map $H : M \to M'$ of class $\mathcal{C}^{n'-n+1}$.*

Then

$$H\left(\left(\operatorname{sing\,supp}_{\mathcal{C}^\infty} H\right)^\circ\right) \subseteq \mathcal{I}_{M'}.$$

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Thank you for your attention!