Random entire functions in linear dynamics

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une vidéo pour Pepe

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This has motivated our work.

Wiman-Valiron theory

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \mathbb{C}$$

be an entire function.

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A finer study of the relationship between the growth of f and the coefficients a_n is undertaken in the Wiman-Valiron theory (1914-18).

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One of the corner-stones of the Wiman-Valiron theory is the Wiman-Valiron inequality, which says that, for any $\delta > 0$

$$\max_{|z|=r} |f(z)| \leq \mu_f(r)\big(\log \mu_f(r)\big)^{\frac{1}{2}+\delta}$$

for all $r \geq 0$ outside some small exceptional set E, where

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Here, the set $E \subset [0, \infty)$ is of finite logarithmic measure, that is

$$\int_{E\cap[1,\infty)} \frac{1}{r} dr < \infty.$$

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and hence, for any $\delta > 0$,

$$\max_{|z|=r} |f(z)| \leq \mu_f(r) \big(\log \mu_f(r) \big)^{\frac{1}{2}+\delta} \sim e^r r^\delta$$

outside some exceptional set E.

Probabilistic Wiman-Valiron theory

In 1930, Lévy studied random entire functions

$$\sum_{n=0}^{\infty} a_n X_n z^n,$$

where $(X_n)_{n\geq 0}$ is an independent sequence of random variables uniformly distributed on the unit circle $\mathbb{T}$ (Steinhaus variables).

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Then, for any $\delta > 0$, almost surely

$$\max_{|z|=r} \Big| \sum_{n=0}^{\infty} a_n X_n z^n \Big| \leq \mu_f(r)\big(\log \mu_f(r)\big)^{\frac{1}{4}+\delta}$$

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Thus, randomizing the coefficients lowers the exponent $\frac{1}{2}$ in the Wiman-Valiron inequality to $\frac{1}{4}$: Lévy's phenomenon (Kuryliak et al. 2014).

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$$\max_{|z|=r} \left| \sum_{n=0}^{\infty} \frac{X_n}{n!} z^n \right| \leq \frac{e^r}{r^{\frac{1}{4}}} r^{\delta}$$

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Finer methods show that this holds for all large $r > 0$ and with $C\sqrt{\log r}$ instead of r^δ (Nikula 2014).

Probabilistic Wiman-Valiron theory

Now, Rosenbloom (1962) improved the (deterministic) Wiman-Valiron inequality: for any $\delta > 0$,

$$\max_{|z|=r} |f(z)| \leq \mu_f(r)\big(\log \mu_f(r)\big)^{\frac{1}{2}}\big(\log\log \mu_f(r)\big)^{1+\delta}$$

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Independently of Lévy, Erdős and Rényi (1969) showed that if $(X_n)_n$ is independent and uniformly $\{-1,+1\}$-distributed (Rademacher variables) then, for any $\delta > 0$, almost surely

$$\max_{|z|=r} \Big|\sum_{n=0}^{\infty} a_n X_n z^n\Big| \leq \mu_f(r)\big(\log \mu_f(r)\big)^{\frac{1}{4}}\big(\log\log \mu_f(r)\big)^{1+\delta}$$

for all $r \geq 0$ outside some small exceptional set E.

Again, we observe Lévy's phenomenon.

Probabilistic Wiman-Valiron theory

Since the mid-1990's there is a school of mathematicians at Lviv (Ukraine) who work on the Wiman-Valiron inequality and its random versions:

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Question (Skaskiv (Kuryliak 2017))

Does Levi's phenomenon hold in the case of unbounded random variables?

An answer is also essential for our application in linear dynamics.

Main result

As an answer to Skaskiv's question, Kuryliak (2017) extended the Erdős-Rényi inequality to centred subgaussian random variables (which include all bounded variables and all Gaussian variables), however at a certain price:

$$\max_{|z|=r} \Big| \sum_{n=0}^{\infty} a_n X_n z^n \Big| \leq \mu_f(r) \big(\log \mu_f(r) \big)^{\frac{1}{4}} \big(\log \log \mu_f(r) \big)^{\frac{3}{2}+\delta}$$

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Here is our main result:

Theorem (Agneessens, G-E)

Let $(X_n)_{n\geq 0}$ be an i.i.d. sequence of centred subgaussian random variables. Then, for every $\delta > 0$, almost surely,

$$\max_{|z|=r}\Big|\sum_{n=0}^{\infty} a_n X_n z^n\Big| \leq \mu_f(r)\big(\log \mu_f(r)\big)^{\frac{1}{4}}\big(\log\log \mu_f(r)\big)^{1+\delta}$$

outside a set $E \subset [0,\infty)$ of finite logarithmic measure.

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However, we follow Erdős-Rényi in looking first for an estimate involving

$$S_f(r) = \Big(\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \Big)^{1/2}$$

instead of

$$\mu_f(r) = \sup_{n \geq 0} |a_n| r^n.$$

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We have also corresponding results for functions on the unit disk.

Application in Linear Dynamics

Consider the derivative operator on the space $H(\mathbb{C})$ of entire functions:

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'.$$

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• frequently hypercyclic (some orbit meets every open set often) –

here, $f \in H(\mathbb{C})$ is frequently hypercyclic if, for any non-empty open set $U \subset H(\mathbb{C})$,

$$\underline{\text{dens}}\{n \geq 0 : D^n f \in U\} > 0.$$

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Drasin and Saksman (2012) showed that there is a frequently hypercyclic entire function f for D with

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and that's optimal (Blasco, Bonilla, G-E 2010). Nikula (2014) obtained the Drasin-Saksman result with an additional factor of $\sqrt{\log r}$.

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Now, note that D is a weighted shift operator:

$$D : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n.$$

Application in Linear Dynamics

More generally, consider any weighted shift operator on $H(\mathbb{C})$:

$$B_w : \sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} w_{n+1} a_{n+1} z^n.$$

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It is well known that B_w is chaotic if and only if

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is an entire function. Randomizing this sequence gives us frequently hypercyclic vectors!

Theorem (Agneessens, G-E)

Let B_w be a chaotic weighted shift on $H(\mathbb{C})$ and $(X_n)_{n\geq 0}$ an i.i.d. sequence of subgaussian random variables of full support. Then

$$\sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} z^n$$

is almost surely frequently hypercyclic for B_w.

Corollary (Agneessens, G-E)

Let B_w be a chaotic weighted shift on $H(\mathbb{C})$ and $(X_n)_{n\geq 0}$ an i.i.d. sequence of centred subgaussian random variables of full support. Then

$$\sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} z^n$$

is almost surely frequently hypercyclic for B_w. Moreover, for every $\delta > 0$, almost surely,

$$\max_{|z|=r} \Big| \sum_{n=0}^{\infty} \frac{X_n}{w_1 \dots w_n} z^n \Big| \leq \mu_f(r) \big(\log \mu_f(r) \big)^{\frac{1}{4}} \big(\log \log \mu_f(r) \big)^{1+\delta}$$

outside a set $E \subset [0, \infty)$ of finite logarithmic measure such that.

Here, $f(z) = \sum_{n=0}^{\infty} \frac{1}{w_1 \dots w_n} z^n$.

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For particular weighted shifts, growth rates for all large r are known (Bernal-Bonilla, Agneessens,. . .)

- K. Agneessens, Frequently hypercyclic random vectors, *Proc. Amer. Math. Soc.* 151 (2023), 1103–1117.
- K. Agneessens, Rate of growth of frequently hypercyclic random functions for weighted shifts, to appear in: *Complex Anal. Oper. Theory*
- K. Agneessens and K.-G. Grosse-Erdmann, Rate of growth of random analytic functions, with an application to linear dynamics, to appear in: *Canad. J. Math.*
- K.-G. Grosse-Erdmann, A note on the Wiman-Valiron inequality, *Arch. Math. (Basel)* 124 (2025), 63–74.
- P. Erdős and A. Rényi, On random entire functions, *Zastos. Mat.* 10 (1969), 47–55.
- A. Kuryliak, Subnormal independent random variables and Levy's phenomenon for entire functions, *Mat. Stud.* 47 (2017), no. 1, 10–19.
- P. Lévy, Sur la croissance des fonctions entières, *Bull. Soc. Math. France* 58 (1930), 127–149.

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