Bohr compactification and Chu duality of non-abelian locally compact groups

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Presented at "Workshop on Functional Analysis".
Conference in honour of Professor José Bonet on the occasion of his 70th Birthday
June 17-19, 2025, Valencia (Spain)

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Joint work with María V. Ferrer

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Pontryagin-van Kampen duality

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Theorem [Pontryagin, 1934; van Kampen, 1935/6]

If G is locally compact and Abelian, $\widehat{G}$ is also locally compact and Abelian and $G \cong \widehat{\widehat{G}}$.

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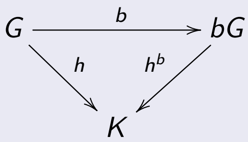
Pontryagin-van Kampen Theorem and, with more generality, duality methods are very effective the study of topological abelian groups.

Bohr Compactification

Definition

The Bohr compactification of an arbitrary topological group G is a pair (bG, b) where bG is a compact Hausdorff group and b is a continuous homomorphism from G onto a dense subgroup of bG with the following universal property:



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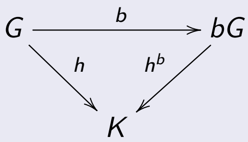

The group bG is called the *Bohr compactification* of G.

The topology that G receives from bG is called *Bohr topology*.

Bohr Compactification

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(1) Consider the commutative Banach algebra with unity $(AP(G), \|\cdot\|_\infty)$. Then bG is the Gel'fand space associated to this algebra.

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(2) Take the continuous homomorphisms

$$h : G \longrightarrow K_h \quad (K_h \text{ is a compact group})$$

and the evaluation mapping

$$e : G \hookrightarrow \prod_{h\in\mathcal{H}} K_h$$

(taking care of the cardinality of $\mathcal{H}$)
then

$$bG \cong cl_{\prod K_h} e(G)$$

Bohr compactification

(3) Finally, we may define the supreme of all precompact topologies on G which are coarser than the original topology of the group. This equips G with a precompact topology (the Bohr topology) whose completion is bG.

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Definition

A topological group G is said to be a maximally almost periodic (MAP) group if the Bohr homomorphism $b : G \to bG$ is injective. This means: Whenever $g \in G$ and $g \neq e_G \in G$, there exists a continuous homomorphism ϕ of G into a compact group, say K_g, such that $\phi(g) \neq e \in K_g$.

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An Abelian topological group is MAP if and only if whenever $g \in G$ and $g \neq 0 \in G$, there exists a continuous homomorphism $\phi : G \to \mathbb{T}$ such that $\phi(g) \neq 0 \in \mathbb{T}$.

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The Gel'fand-Raĭkov Theorem implies that locally compact Abelian groups (LCAGs) are MAP

Bohr compactification

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The Bohr topology of an Abelian discrete group G can be realized as the weak topology induced on G by the group $\mathrm{Hom}(G, \mathbb{T})$ of homomorphisms from G into the usual circle group $\mathbb{T}$.

It coincides with the largest totally bounded group topology of the group.

Motivation

Suppose an algebraic group G is equipped with two locally compact topologies $G_1 = (G, \tau_1)$ and $G_2 = (G, \tau_2)$. In case G is abelian,we have that G_1 and G_2 are (naturally) isomorphic if and only if so are their respective Bohr compactifications bG_1 and bG_2.

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Therefore the Bohr compactification of a locally compact abelian group completely characterizes its topological and algebraic structure.

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Therefore the Bohr compactification of a locally compact abelian group completely characterizes its topological and algebraic structure.

It is known that this fact do not extend to non abelian groups and basically every option is possible for these groups.

Here we are interested in studying to what extent the Bohr compactification of a non abelian locally compact group reflects its topological and algebraic structure.

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Example (Heisenberg group)

Let H be the Heisenberg integral group

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

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We have: for all x, y in H, the sequence $\{[x^{n!}, y^{n!}]\}$ converges to the neutral element in the Bohr topology.

non-Abelian groups

Example (lamplighter group)

Let G be $(\sum_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ (the group is known as the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$).

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Then bG is topologically isomorphic to the group

$$\overline{\sum_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}}^{\,bG} \rtimes b\mathbb{Z}$$

where the group $\overline{\sum_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}}^{\,bG}$ is metrizable. Therefore, there are many non-trivial Bohr convergent sequences in G.

Another example of this sort is due to Moran

Example (Moran, 1971)

Let $\{p_i\}$ be an infinite sequence of distinct prime numbers ($p_i > 2$), and let F_i be the projective special linear group of dimension two over the Galois field $GF(p_i)$ of order p_i. If $G = \sum_{i\in\mathbb{N}} F_i$ then $bG_d = \prod_{n\in\mathbb{N}} F_n$.

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That is, compact groups G such that $bG_d = G$.
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Many more examples and properties of these groups were found by Hart and Kunen (2011).

A unitary representation σ of the (topological) group G is a (continuous) homomorphism into the group of all linear isometries of a complex Hilbert space $\mathcal{H}^\sigma$, the so called unitary group $\mathbb{U}(\mathcal{H}^\sigma)$.

Here, the unitary group $\mathbb{U}(\mathcal{H}^\sigma)$ is equipped with the weak (equiv., strong) operator topology. Hence σ is continuous if for every $u, v \in \mathcal{H}^\sigma$, the matrix coefficient function $g \mapsto \langle \sigma(g)u, v \rangle$ is a continuous map of G into the complex plane.

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Unitary representations of locally compact groups

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An irreducible representation (irrep) of G is a continuous unitary representation σ such that $\operatorname{ran}(\sigma)$ is irreducible.

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Two unitary representations $\sigma : G \to U(\mathcal{H}^\sigma)$ and $\rho : G \to U(\mathcal{H}^\rho)$ are equivalent $(\rho \sim \psi)$ if there exists a unitary operator $M : \mathcal{H}^\sigma \to \mathcal{H}^\rho$ such that $\rho(x) = M^{-1}\psi(x)M$ for all $x \in G$.

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In this talk we shall just be concerned with finite dimensional representations.

Denote by $\mathrm{rep}_n(G)$ the set of all continuous n-dimensional unitary representations of a topological group G, i. e., the set of all continuous homomorphisms of G into the unitary group $\mathbb{U}(n)$, equipped with the compact-open topology.

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Set $G_n^x = \frac{\mathrm{rep}_n(G)}{\sim}$ as the set of equivalence classes of unitary representations of dimension n equipped with the quotient topology. The partial dual of G is defined by $\widehat{G}_n = \frac{\mathrm{irrep}_n(G)}{\sim}$.

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Set $G_n^\times = \frac{\mathrm{rep}_n(G)}{\sim}$ as the set of equivalence classes of unitary representations of dimension n equipped with the quotient topology. The partial dual of G is defined by $\widehat{G}_n = \frac{\mathrm{irrep}_n(G)}{\sim}$.

It is easily seen that if G is a compact group then each partial dual space $G_n^\times$ is uniformly discrete.

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Characterization of self-bohrifying groups

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Proposition

Let G be a compact group and let G_d be the same algebraic group equipped with the discrete topology. The following assertions are equivalent:

1. G_d is tall.
2. G is self-bohrifying.
3. $\widehat{(G_d)}_n \cup \{1_G\}$ is discrete for all $n \in \mathbb{N}$.
4. G is a *vdW*-group (every group homomorphism of G into a compact group is continuous).

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Unitary (Chu) duality

Recall that if $A : \mathbb{C}^m \to \mathbb{C}^m$ and $B : \mathbb{C}^n \to \mathbb{C}^n$ are linear operators, then $A \oplus B : \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ denotes the *direct sum*, and $A \otimes B : \mathbb{C}^{mn} \to \mathbb{C}^{mn}$ denotes the *tensor product*, of A and B, respectively.

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Thus, we can define the following operations on $\mathrm{rep}(G)$

1. $(D \oplus D')(x) = D(x) \oplus D'(x)$, $D, D' \in \mathrm{rep}(G)$ and $x \in G$;
2. $(D \otimes D')(x) = D(x) \otimes D'(x)$, $D, D' \in \mathrm{rep}(G)$ and $x \in G$;
3. $(U^{-1}DU)(x) = U^{-1}D(x)U$, $D \in \mathrm{rep}_n(G)$, $U \in \mathbb{U}(n)$ and $x \in G$.

Unitary (Chu) duality

A quasi-representation of G is a mapping $Q : \mathrm{rep}(G) \longrightarrow \mathbb{U}$ with the following properties:

1. $Q[\mathrm{rep}_n(G)] \subset \mathbb{U}(n)$;
2. $Q(D \oplus D') = Q(D) \oplus Q(D'),\ D, D' \in \mathrm{rep}(G)$;
3. $Q(D \otimes D') = Q(D) \otimes Q(D'),\ D, D' \in \mathrm{rep}(G)$;
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The set of all *continuous* quasi-representations of G equipped with the compact-open topology is a topological group with pointwise multiplication as composition law, called the Chu quasi-dual group (bidual group, for short) of G and denoted by G^{xx}.

Unitary (Chu) duality

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Definition

We say that the locally compact group G satisfies Chu duality when the evaluation mapping $\mathcal{E}$ is an isomorphism of topological groups (Chu, 1966).

Chu quasi dual versus Bohr compactification

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Theorem (H. and Wu, 2006)

Let $G = \sum_{i \in I} F_i$, where F_i is a finite simple non-abelian group for each $i \in I$. Then the group G is Chu if and only if the set $\{exp(F_i) : i \in I\}$ is bounded.

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Theorem (A. Thom, 2013)

A finitely generated group satisfies Chu duality if, and only if, it is virtually abelian.

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New results

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Theorem (Ferrer and H., 2025))

Let H be a finite simple non abelian group, let I be an infinite index set and consider the discrete group $G = H^{(I)}$. Then G is Chu reflexive and $\widehat{G}_n$ is a discrete topological space for all $n \in \mathbb{N}$.

New results

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Theorem (Ferrer and H., 2025)

Let G be a discrete group. Then G^{xx} is topologically isomorphic to bG if and only if the space $\widehat{G}_n$ is finite for any $n \in \mathbb{N}$.

DEDICATED TO PEPE!

THANK YOU FOR YOUR ATTENTION!