On the Grothendieck property for Banach spaces $Lip_0(M)$ of Lipschitz functions

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6. M. Gonzales, T. Kania, Grothendieck spaces: the landscape and perspectives- survey article.

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6. (CH) There is compact X without $\beta\mathbb{N}$ and such that $C(X)$ is Grothendieck but without quotients isomorphic to ℓ_∞ (Talagrand). The same conclusion without (CH) (Sobota-Zdomskyy).

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7. $((MA) \wedge \sim (CH))$ Every nonreflexive Grothendieck space has a quotient isomorphic to ℓ_∞ (Haydon-Levy-Odell).

1. A series $\sum_n x_n$ in a Banach space E is weakly unconditionally converging if for every $x^* \in E^*$ one has $\sum_n x^*(x_n) < \infty$. An operator $T : E \to F$ is **unconditionally converging** if for each weakly unconditionally converging series $\sum_n x_n$ the series $\sum_n T(x_n)$ is unconditionally converging in F.

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3. **A Banach space E has property (V), whenever every unconditionally converging operator $T : E \to F$ is weakly compact** (Pełczyński).
4. Reflexive spaces and $C(K)$-spaces have property (V).
5. Moreover, property (V) is inherited by quotients (but not by closed subspaces).

1. $C[0,1]$ has property (V) but it is not Grothendieck.
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Theorem 3 (Hájek–Novotný)

M an infinite metric space. Then $Lip_0(M)$ contains an isomorphic copy of $\ell_\infty(d(M))$. Hence $\mathcal{F}(M)$ contains a complemented copy of $\ell_1(d(M))$.

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3. Indeed, the space $\mathcal{F}(M)$ contains a complemented copy of the space $\ell_1(d(M))$. Hence $\mathcal{F}(M)$ contains a complemented copy of ℓ_1.
4. Since complemented subspaces of a Banach space with the Grothendieck property are Grothendieck, and ℓ_1 fails the Grothendieck property, the claim holds.

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3. To prove that $Lip_0(M)$ is not Grothendieck (using Räbiger's criterion), we need either to look for a quotient of $Lip_0(M)$ isomorphic to c_0 or to show that the dual space $Lip_0(M)^*$ is not weakly sequentially complete.

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4. Grothendieck property is preserved by continuous (open) linear surjections, so $Lip_0(E)$ **lacks the Grothendieck property if E^* fails to have it** (since $Lip_0(E)$ contains a complemented copy of E^*). This provides examples of non-Grothendieck spaces $Lip_0(E)$ over Banach spaces E.

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Theorem 4 (Bargetz–J.K–Sobota)

Let E be a Banach space satisfying any of the following conditions: (1) There is a continuous linear surjection $T : E^* \to \ell_1$. (2) E is separable and contains an isomorphic copy of a predual of ℓ_1. (3) E contains a complemented copy of ℓ_1. (4) E has property (V) and E is not Grothendieck. Then $Lip_0(E)$ is not a Grothendieck space.

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Corollary 5

If $E = c_0$ or $E = \ell_1$, then $Lip_0(E)$ is not a Grothendieck space.

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Let E be a separable Banach space not weakly sequentially complete. Then $Lip_0(E)$ is not a Grothendieck space.

2. Indeed, $\mathcal{F}(E)$ contains an isometric copy of E. If E is separable, (by the lifting property) the space $\mathcal{F}(E)$ contains a linear isometric copy of E (Godefrey-Kalton).

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3. Hence $Lip_0(E)^* \simeq \mathcal{F}(E)^{**}$ is not weakly sequentially complete, so Räbiger's theorem applies.

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2. **M is a separable Banach space which is not weakly sequentially complete**, e.g. M is the James space $\mathcal{J}$.
3. **M contains a bilipschitz copy of the unit sphere** S_{c_0}, e.g. $M = C(K)$ for some infinite compact space K.

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4. M **is a net in** c_0 **or** ℓ_1.

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4. **M is a net in c_0 or ℓ_1.**
5. **M is a $C(K)$-space, $L_1(\mu)$-space, $Lip_0(M)$-space, or $\mathcal{F}(M)$-space.**

1. A subspace $\mathcal{N}$ of a metric space M is a *net* if there are $\varepsilon, \delta > 0$ such that $\rho(x, y) \geq \varepsilon$ for every $x \neq y \in \mathcal{N}$ and for every $x \in M$ there is $y \in \mathcal{N}$ with $\rho(x, y) < \delta$. $Lip_0(\mathcal{N})$, where $\mathcal{N}$ is a net in either c_0 or ℓ_1, admits a continuous operator onto ℓ_1 (Candido, Cúth, Doucha)

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3. Banach spaces c_0 and $C(K)$ for K metric compact are absolute Lipschitz retracts (Lindenstrauss).

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Theorem 8 (Bargetz–J.K–Sobota)

Let E be a separable Banach space which is an absolute Lipschitz retract and contains c_0. If M contains a bilipschitz copy of S_E of E, then $Lip_0(M)$ is not Grothendieck (since it admits a continuous operator onto ℓ_1.)

1. Again recall that $Lip_0(E)$ with a continuous linear surjection onto ℓ_1 are not Grothendieck.

Theorem 8 (Bargetz–J.K–Sobota)

Let E be a separable Banach space which is an absolute Lipschitz retract and contains c_0. If M contains a bilipschitz copy of S_E of E, then $Lip_0(M)$ is not Grothendieck (since it admits a continuous operator onto ℓ_1.)

Corollary 9

If M contains a bilipschitz copy of the unit sphere S_{c_0} of c_0, then $Lip_0(M)$ is not Grothendieck.

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2. Such E provides an example of Grothendieck $Lip_0(E)$.

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Is $Lip_0(\mathbb{R}^2)$ a Grothendieck space? Note $Lip_0(\mathbb{R}) \simeq L^\infty(\mathbb{R})$.

3. Note that $Lip_0(\mathbb{R}^n)^*$ $(n \geq 1)$ is not Grothendieck, since it contains $\mathcal{F}(\mathbb{R}^n)$ complemented (Cuth-Kalenda-Kaplicky).

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2. **If $Lip_0(\ell_2)$ is not Grothendieck** and one can find a separable infinite-dimensional Banach space E such that $Lip_0(E)$ is Grothendieck, then this would answer in negative a question (Candido, Cúth, Doucha) whether $Lip_0(\ell_2)$ is complemented in $Lip_0(F)$ for every separable infinite-dimensional Banach space F.