Linear topological invariants for kernels of differential operators

Thomas Kalmes

Partly joint work with A. Debrouwere (Vrije Universiteit Brussel, Belgium)

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Equip $C^\infty(X)$ with its usual Fréchet space topology: $(K_n)_{n\in\mathbb{N}}$ be a compact exhaustion of X

$$\forall\, n\in\mathbb{N}\,:\,\|g\|_n:=\max_{|\alpha|\leq n, x\in K_n}|D^\alpha g(x)|\quad (g\in C^\infty(X))$$

$\Rightarrow (\|\cdot\|_n)_{n\in\mathbb{N}}$ increasing sequence of seminorms on $C^\infty(X)$, $P(D)$ continuous linear operator on $C^\infty(X)$

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Then,

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$$F(\Lambda; E) \to F(\Lambda; E), (x_\lambda)_{\lambda\in\Lambda} \mapsto (T(x_\lambda))_{\lambda\in\Lambda} = \Phi^{-1} \circ T\varepsilon\mathrm{id}_{F(\Lambda)} \circ \Phi\left((x_\lambda)_{\lambda\in\Lambda}\right)$$

Parameter dependence problem w.r.t. F:
Let $P(D) : C^\infty(X) \to C^\infty(X)$ be surjective; for which LCS F is the mapping $P(D)\varepsilon\mathrm{id}_F : C^\infty(X)\varepsilon F \to C^\infty(X)\varepsilon F$ surjective?

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What if F is the dual space of a Fréchet space (e.g. $F \in \{\mathscr{S}'(\mathbb{R}^d), \mathscr{E}'(\Lambda)\}$), or if $F \in \{\mathscr{A}(\Lambda), \mathscr{D}'(\Lambda)\}$?

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- examples: $\mathscr{S}(\mathbb{R}^d)$, $\mathcal{O}(\mathbb{C}^d)$; non-examples: $C^\infty(X)$, $\mathcal{O}(\mathbb{D}^d)$

Theorem [6, Vogt-Wagner, 1975-1980]

E be nuclear Fréchet, $T : E \to E$ be surjective, linear, continuous. TFAE

(i) $T\varepsilon\mathrm{id}_{\mathscr{S}'(\mathbb{R})} : E\varepsilon\mathscr{S}'(\mathbb{R}) \to E\varepsilon\mathscr{S}'(\mathbb{R})$ is surjective.

(ii) $\ker T$ satisfies (Ω).

(iii) $T\varepsilon\mathrm{id}_F : E\varepsilon F \to E\varepsilon F$ is surjective for every LCS $F \cong \prod_{i\in I} F_i$ with F_i a Montel (DF)-space such that its strong dual space $(F_i)_b'$ has (DN).

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Schwartz kernel theorem: $\mathscr{D}'(X)\varepsilon\mathscr{D}'(\mathbb{R}) \cong \mathscr{D}'(X \times \mathbb{R})$, and $P(D)\varepsilon\mathrm{id}_{\mathscr{D}'(\mathbb{R})}$ is $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$, with $P^+(\xi_1, \ldots, \xi_d, \xi_{d+1}) = P(\xi_1, \ldots, \xi_d)$

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$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective $\overset{?}{\Rightarrow}$ $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ surjective?

Malgrange: for hypoelliptic $P(D)$ it holds $C_P^\infty(X) = \mathscr{D}'_P(X)$ as LCS and $P(D) : C^\infty(X) \to C^\infty(X)$ surjective iff $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ surjective

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Theorem [3, 4, 5, K. 2012, 2019]

(a) For $d \geq 3$ there is a hypoelliptic $P(D)$ and a (non-convex) set X such that $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective and $\mathscr{D}'_P(X)$ does not satisfy $(P\Omega)$; so, $P(D) : C^\infty(X) \to C^\infty(X)$ is surjective and $C_P^\infty(X)$ does not have (Ω).

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- (b-i) $P(\xi_1, \ldots, \xi_d) = Q(\xi_1, \ldots, \xi_k)$ with $1 \leq k \leq d$ and Q elliptic.
- (b-ii) $P(D)$ is semi-elliptic with a single characteristic direction, e.g. a parabolic operator.
- (b-iii) $P(D)$ factorizes into first order operators.
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For all operators in (b), surjectivity of $P(D)$ on $\mathscr{D}'(X)$ is equivalent to surjectivity on $C^\infty(X)$!

Theorem [2, Debrouwere, K. 2023]

(a) Let $P(D) : C^\infty(X) \to C^\infty(X)$ be surjective. If $\mathscr{D}'_P(X)$ has $(P\Omega)$, then $C^\infty_P(X)$ has (Ω).

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Corollary [2, Debrouwere, K. 2023]

Let $P(D): C^\infty(X) \to C^\infty(X)$ be surjective. Then, $C^\infty_P(X)$ satisfies (Ω) if

(i) X is convex.

(ii) $P(\xi_1, \ldots, \xi_d) = Q(\xi_1, \ldots, \xi_k)$ with $1 \leq k \leq d$ and Q elliptic.

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Thank you for your attention!

HAPPY BIRTHDAY, PEPE!!!