C-maximal regularity and C-admissibility for semigroups on locally convex spaces

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C-maximal regularity & C-admissibility

Let X be a sequentially complete lcHs, $r>0$ and $f \in C([0,r];X)$.

Consider the abstract Cauchy problem

$$\begin{aligned} u'(t) &= A u(t) + f(t), \quad t \in [0,r], \\ u(0) &= x_0 \in X, \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} u'(t) - A u(t) &= f(t), \quad t \in \dots \\ u(0) &= x_0 \end{aligned}$$

where A is the generator of a semigroup $(T(t))_{t\geq 0}$ in $\mathcal{L}(X)$ which is

- strongly continuous, i.e. the map $[0,\infty) \to X$, $t \mapsto T(t)x$, is continuous for each $x \in X$,
- locally equicontinuous, i.e. $\forall q \in \Gamma_X$, $t_0 \geq 0$ $\exists p \in \Gamma_X$, $C \geq 0$ $\forall t \in [0,t_0], x \in X$: $q(T(t)x) \leq C p(x)$.

Looking at the mild solution of the ACP

$$u(t) = T(t)x_0 + \underbrace{\int_0^t T(t-s) f(s)\, ds}_{=: (T*f)(t)}, \quad t \in [0,r],$$

we say that $(T(t))_{t\geq 0}$ is C-maximal regular (for r) if

(i) $(T*f)(t) \in D(A)$ for all $t \in [0,r]$, $\Longleftrightarrow:$ $(T(t))_{t\geq 0}$ is C-admissible.

(ii) $A(T*f) \in C([0,r];X)$

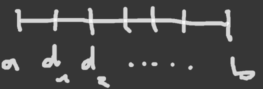
for all $f \in C([0,r];X)$.

Question

C-adm. $\Rightarrow$ C-max. reg.?

Bounded semivariation

Let $\alpha : [a,b] \to \mathcal{L}(X)$, $\underbrace{a = d_0 < d_1 < \ldots < d_n = b}_{=:d}$, $p, q \in \Gamma_X$ and



$$SV_{q,p,d}(\alpha) := \sup \left\{ q\Big(\sum_{i=1}^{n} (\alpha(d_i) - \alpha(d_{i-1}))x_i \Big) \;\Big|\; x_i \in X, p(x_i) \le 1 \right\}.$$

as well as

$$SV_{q,p}(\alpha) := \sup_{d \in D[a,b]} SV_{p,q,d}(\alpha).$$

- semivariation
- set of all partitions of $[a,b]$

We say that α is of bounded semivariation on $[a,b]$ if

$$\forall q \in \Gamma_X \; \exists p \in \Gamma_X : SV_{q,p}(\alpha) < \infty.$$

Proposition (Hönig 1973) α of bounded semivariation and $f \in C([a,b];X)$

$\Rightarrow$ f Riemann-Stieltjes integrable w.r.t. α & $\forall q \in \Gamma_X \; \exists p \in \Gamma_X : q\Big(\int_a^b f(s)\, d\alpha(s) \Big) \le SV_{q,p}(\alpha) \sup_{s \in [a,b]} p(f(s))$

Let $r > 0$. We say that an sg $(T(t))_{t \ge 0}$ in $\mathcal{L}(X)$ is of bounded semivariation if the map $[0,r] \to \mathcal{L}(X)$, $t \mapsto T(t)$, is of bounded semivariation.

Main theorem (KS 2024, Travis 1981 for Banach spaces X)

The following assertions are equivalent if X is a C-space:

(a) $(T(t))_{t\geq 0}$ is C-maximal regular, i.e. $(T*f)(t)\in D(A)$ $\forall t\in[0,r]$ & $A(T*f)\in C([0,r];X)$ $\forall f\in C([0,r];X)$.

(b) $(T(t))_{t\geq 0}$ is C-admissible.

(c) $(T(t))_{t\geq 0}$ is of bounded semivariation, i.e. $\forall q\in\Gamma_X\ \exists p\in\Gamma_X:\ SV_{p,q}(T_{|[0,r]})<\infty$.

Proof. (a) $\Rightarrow$ (b): ✓

(c) $\Rightarrow$ (a): (i) Show that $(T*f)(t)=\int_0^t T(t-s)f(s)ds\in D(A)$ and $A\int_0^t T(t-s)f(s)ds=\int_0^t f(s)\,dT(t-s)$.

Idea $g_n(s):=\begin{cases} T(t-s)f(\frac{i}{n}t), & \frac{i-1}{n}t<s\leq\frac{i}{n}t \\ T(t)f(0), & s=0\end{cases}$, and show that $\int_0^t g_n(s)ds\to(T*f)(t)$,

$$A\int_0^t g_n(s)ds\to\int_0^t f(s)\,dT(t-s).$$

(ii) Show that the map $[0,r]\to X$, $t\mapsto\int_0^t f(s)\,dT(t-s)$, is continuous.

(b) $\Rightarrow$ (c): Consider

$\Psi: C([0,r];X)\to X$, $\Psi(f):=(T*f)(r)$, linear + continuous.

$\overset{(b)}{\Rightarrow}$ $A\Psi: C([0,r];X)\to X$, $A\Psi(f)=A(T*f)(r)$, linear + closed.

X C-space :$\Leftrightarrow$ Any closed linear operator $C: C([0,r];X)\to X$ is continuous.

Examples (KS 2024, Travis 1981 for Banach spaces X)

(i) X a Fréchet space or a strong dual of a reflexive Fréchet space & $(T(t))_{t\geq 0}$ a strongly continuous sg on X with generator $A \in \mathcal{L}(X)$.

$\Rightarrow (T(t))_{t\geq 0}$ C-maximal regular.

(ii) $X := (c_0, \|\cdot\|_\infty)$ & $T(t)x := (e^{-nt}x_n)_{n\in\mathbb{N}}$, $t\geq 0$, $x=(x_n)_{n\in\mathbb{N}} \in c_0$.

$\Rightarrow (T(t))_{t\geq 0}$ C-maximal regular & $D(A) = \{x \in c_0 \mid (-nx_n) \in c_0\} \neq c_0$.

(iii) $X := (\ell^\infty, \mu(\ell^\infty, \ell^1))$ & $T(t)x := (e^{-nt}x_n)_{n\in\mathbb{N}}$, $t\geq 0$, $x=(x_n)_{n\in\mathbb{N}} \in \ell^\infty$.

$\Rightarrow (T(t))_{t\geq 0}$ C-maximal regular & $D(A) = \{x \in \ell^\infty \mid (-nx_n) \in \ell^\infty\} \neq \ell^\infty$.

Remark (Cooper 1978)

- $(\ell^\infty, \mu(\ell^\infty, \ell^1))$ is not a Fréchet space (or the strong dual of a reflexive one).
- The Mackey topology $\mu(\ell^\infty, \ell^1)$ is induced by the system of seminorms

$$p_a(x) := \sup_{n\in\mathbb{N}} |x_n| a_n, \quad x=(x_n) \in \ell^\infty, \quad a=(a_n) \in c_0^+.$$

- $\mu(\ell^\infty, \ell^1) = \gamma(\|\cdot\|_\infty, \tau_{co})$ is a mixed topology.

C-spaces

Recall

An lcHs X is called a C-space if for all $r>0$ any closed linear map $B: C([0,r];X) \to X$ is continuous.

Proposition (KS 2024)

(a) X Fréchet or a strong dual of a reflexive Fréchet space $\Rightarrow$ X C-space.

(b) $C([0,r];X)'$ weakly sequentially complete for all $r>0$ & X separable B_r-complete Mackey space

$\Rightarrow$ X C-space.

Proof. (a) Closed graph theorem of de Wilde / Pták (de Wilde 1971, Pták 1958)

(b) Closed graph theorem of Kalton (Kalton 1971)

& $C([0,r];X) \simeq \underbrace{C([0,r]) \varepsilon X}_{:= L_e(C([0,r])'_{\kappa}, X) \ni u \mapsto [x \mapsto u(\delta_x)]}$ is a Mackey space (Bierstedt 1973, Kerjean 2018).

Example $(\ell^\infty, \mu(\ell^\infty, \ell^1))$ fulfils the conditions of (b).

Open problems

Let Ω be a Polish space & consider the mixed topology $\gamma(\|\cdot\|_\infty, \tau_{co})$ on $C_b(\Omega)$ given by

$$p_v(f) := \sup_{x\in\Omega} |f(x)|v(x), \quad f\in C_b(\Omega), \quad v \text{ non-negative, bounded \& vanishing at } \infty.$$

Question 1 Is $(C_b(\Omega), \gamma(\|\cdot\|_\infty, \tau_{co}))$ a C-space?

Remark 1 (Summers 1972, Cooper 1978)

$(C_b(\Omega), \gamma(\|\cdot\|_\infty, \tau_{co}))$ is a complete separable Mackey space.

Remark 2 (KS 2024)

$X := (C_b(\Omega), \gamma(\|\cdot\|_\infty, \tau_{co})) \implies C([0,r];X)'$ weakly sequentially complete for all $r>0$.

Proof. $C([0,r];X) \simeq (C_b([0,r]\times\Omega), \gamma)$ Bierstedt 1974

Question 2 Is $(C_b(\Omega), \gamma(\|\cdot\|_\infty, \tau_{co}))$ B_r-complete ?

Recall An lcHs X is called B_r-complete if every $\sigma(X',X)$-dense $\sigma^f(X',X)$-closed linear subspace of X' coincides with X' where $\sigma^f(X',X)$ is the finest topology on X' coinciding with $\sigma(X',X)$ on all equicontinuous subsets of X'.

Remark 3
- Q2: Yes if Ω is discrete (Collins 1968) or compact.
- Even in the case $\Omega = \mathbb{R}$ answer to Q2 is not known yet (Summers 1970).

Bedankt voor jullie aandacht!

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Pepe, gefeliciteerd met je verjaardag!