On sequences preserving $q-$Gevrey asymptotic expansions

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Joint work with S. Michalik

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Outline of the talk

Motivation

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The classical theory

A formal power series $\hat{u}(t) = \sum_{p\geq 0} a_p t^p \in \mathbb{C}[[t]]$ is said to be k-summable along direction $d \in \mathbb{R}$ if its Borel transformation of order $1/k$,

$$\mathcal{B}_{(\Gamma(1+p/k))_{p\geq 0}}\left(\sum_{p\geq 0} a_p t^p\right) = \sum_{p\geq 0} \frac{a_p}{\Gamma\left(1+\frac{p}{k}\right)} t^p$$

defines an analytic function on some neighborhood of the origin, which can be analytically extended to a function u_d defined on an infinite sector of bisecting direction d, say S_d, with exponential growth of order k at infinity in that sector, i.e. for every $\tilde{S}_d \prec S_d$ there exist $A, B > 0$ such that

$$|u_d(t)| \leq A e^{B|t|^k}, \qquad t \in \tilde{S}_d.$$

A Laplace transform can be applied to u_d along direction d.

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Formal solution: $\hat{u}(t) = \sum_{p \geq 0} a_p t^p \in \mathbb{C}[[t]]$

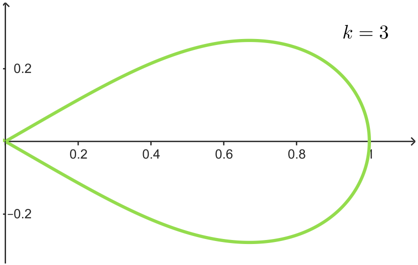
Summability process: Borel-Laplace procedure

Analytic solution: $u_d(t)$

Motivation

Nevanlinna's Theorem: The following statements are equivalent:

- A formal power series $\hat{u}(t) = \sum_{p \geq 0} u_p t^p$ is k-summable along direction $d \in \mathbb{R}$
- There exists a unique holomorphic function u defined on a sectorial region G with bisecting direction d and opening π/k which admits $\hat{u}$ as its Gevrey asymptotic expansion of order $1/k$ in that sectorial region, i.e. for all $\tilde{S}_d \prec G$ there exist $C, A > 0$ s.t.



$$\left| u(t) - \sum_{p=0}^{N} u_p t^p \right| \leq CA^N \Gamma(1+N/k)|t|^{N+1},$$

for all $N \geq 0$ and $t \in \tilde{S}_d$.

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In [1], K. Ichinobe and S. Michalik defined the notion of sequences preserving summability.

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A sequence $m = (m_p)_{p\geq 0}$ of positive real numbers, with $m_0 = 1$, preserves summability if for every $k > 0$, $d \in \mathbb{R}$ and every formal power series $\hat{u} \in \mathbb{C}[[t]]$ the following statements are equivalent:

- $\hat{u}$ is k-summable along direction d.
- $\mathcal{B}_m \hat{u}$ is k-summable along direction d.

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Main motivation

A formal solution of a problem (also the problem itself) can be substituted by some other, which might be more easy to handle, to check summability of the formal solution.

Previous results

The set of sequences preserving summability forms a group with the operation

$$(m_{1,p})_{p\geq 0}\cdot(m_{2,p})_{p\geq 0}=(m_{1,p}m_{2,p})_{p\geq 0}.$$

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Some examples:

- The sequence $(1)_{p\geq 0}$ preserves summability,
- for every $A > 0$, the sequence $(A^p)_{p\geq 0}$ preserves summability,
- Given two sequences of moments associated to kernel functions for generalized summability, of the same positive order, then its quotient sequence preserves summability.

Previous results

Let $0 < q < 1$. We consider the sequence of $q-$factorials $m_q = ([p]_q!)_{p \geq 0}$ defined by $[0]_q! = 1$ and for all $p \geq 1$, $[p]_q! = [p]_q \cdot [p-1]_q \cdots [1]_q$, with

$$[j]_q = 1 + q + \cdots + q^{j-1} = \frac{1-q^j}{1-q}, \quad j \geq 1.$$

The $q-$difference operator is given by

$$\partial_{q,t}(\hat{u}) = \frac{\hat{u}(qt) - \hat{u}(t)}{qt - t},$$

for every $\hat{u} \in \mathbb{C}[[t]]$, if $q \in [0,1)$.

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The main result in [1] reads as follows:

Theorem

The sequence $([p]_q!)_{p\geq 0}$ preserves summability for every $q \in [0,1)$.

Previous results

As an application, the summability of formal solutions to functional equations is characterized in terms of certain simplified equations.

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$$\left\{ \begin{array}{c} P(\partial_{q,t}, \partial_z)u = 0 \\ \partial_{q,t}^j u(0, z) = \varphi_j(z) \in \mathcal{O}(D), \\ j = 0, \ldots, p-1, \end{array} \right. \qquad \left\{ \begin{array}{c} P(\partial_{0,t}, \partial_z)v = 0 \\ \partial_{0,t}^j v(0, z) = \varphi_j(z) \in \mathcal{O}(D), \\ j = 0, \ldots, p-1, \end{array} \right.$$

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$\hat{u}(t,z)$ in the form $\sum_{p\geq 0}\frac{u_p(z)}{[p]_q!}t^p$ in $\mathcal{O}(D)[[t]]$ is a formal solution. $\Leftrightarrow$ $\hat{v}(t,z)$ in the form $\sum_{p\geq 0}u_p(z)t^p$ in $\mathcal{O}(D)[[t]]$ is a formal solution.

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Levels of approximation

As an application, the summability of formal solutions to functional equations is characterized in terms of certain simplified equations.

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$\hat{u}(t, z)$ is a formal solution of Gevrey order $1/k > 0$. $\Leftrightarrow$ $\hat{v}(t, z)$ is a formal solution of Gevrey order $1/k > 0$.

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$\hat{u}(t, z)$ is a formal solution which is k-summable along direction $d \in \mathbb{R}$. $\Leftrightarrow$ $\hat{v}(t, z)$ is a formal solution which is k-summable along direction $d \in \mathbb{R}$.

Previous results

Different theories involving $q-$Gevrey asymptotic expansions have been developed in the last decades.

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The study deals with:

- Preservation of $q-$Gevrey order
- Preservation of $q-$Gevrey asymptotic expansions

Sequences preserving $q-$Gevrey asymptotic expansions

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Let $q>1$. Let S be a sector with vertex at the origin. We say that $f \in \mathcal{O}(S,\mathbb{E})$ admits $\hat{f}(t)=\sum_{p\geq 0} f_p t^p \in \mathbb{E}[[t]]$ as its $q-$Gevrey asymptotic expansion of order $s>0$ (at the origin) if for every $\tilde{S} \prec S$ there exist $C, A>0$ such that

$$\left\| f(t)-\sum_{p=0}^{N} f_p t^p \right\|_{\mathbb{E}} \leq C A^N q^{s\frac{N(N+1)}{2}} |z|^{N+1},$$

for all $N\geq 0$ and all $z\in\tilde{S}$.

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One of the $q-$analogs which provide an analytic meaning to a formal power series is the following, initially developped for $s=1$ in [2].

[2] C. Zhang, Transformations de $q-$Borel$-$Laplace au moyen de la fonction thêta de Jacobi, C. R. Acad. Sci. Paris, Sér. I, Math. 331, No. 1 (2000), 31–34.

Sequences preserving $q-$Gevrey asymptotic expansions

Let $\hat{f} \in \mathbb{E}[[t]]$ such that $g = \mathcal{B}_{q;s}(\hat{f}) \in \mathbb{E}\{t\}$ for some $s > 0$, where

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Moreover, assume that g can be analytically extended to an infinite sector S_d of bisecting direction $d \in \mathbb{R}$ and the extension is of $q-$exponential growth of order $1/s$, i.e. there exist $C, h > 0$ and $\alpha \in \mathbb{R}$ such that

$$\|f(z)\|_{\mathbb{E}} \leq C \exp\left(\frac{\log^2(|z| + h)}{2s\log(q)}\right)(|z| + h)^{\alpha}, \quad z \in S_d.$$

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Then, for every $\gamma \in \mathbb{R}$ such that $L_\gamma = (0,\infty)e^{i\gamma} \subseteq S_d$ the function

$$\mathcal{L}^{\gamma}_{q;s}(g)(t) = \frac{1}{\pi_{q^s}} \int_{L_\gamma} \frac{f(u)}{\Theta_{q^s}\left(\frac{u}{t}\right)} \frac{du}{u}, \quad \text{where } \Theta_{q^s}(z) = \sum_{p\in\mathbb{Z}} \frac{1}{q^{s\frac{p(p-1)}{2}}} z^p,$$

admits $\hat{f}$ as its $q-$Gevrey asymptotic expansion of order s on some finite sector of bisecting direction d and opening $> 2\pi$.

Sequences preserving $q-$Gevrey asymptotic expansions

As a $q-$analog of sequences preserving summability, we defined in [3]:

A sequence m is said to preserve $q-$Gevrey asymptotic expansions if for every $s>0$, $d\in\mathbb{R}$ and $\hat{u}\in\mathbb{E}[[t]]$ the following statements turn out to be equivalent:

(i) $\mathcal{B}_{q;s}(\hat{u})\in\mathbb{E}\{t\}$, and this function can be extended on an infinite sector of bisecting direction d with $q-$exponential growth of order $1/s$ on such sector.

(ii) $\mathcal{B}_{q;s}\mathcal{B}_m(\hat{u})\in\mathbb{E}\{t\}$, and this function can be extended on an infinite sector of bisecting direction d with $q-$exponential growth of order $1/s$ on such sector.

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Remark: Observe the absence of unicity on the assignment.

Main results

A more easy to handle characterization for a sequence to preserve $q-$Gevrey asymptotic expansions is available.

Theorem

A sequence $m = (m_p)_{p\geq 0}$ preserves $q-$Gevrey asymptotic expansions if and only if for every $s > 0$ and every $\theta \neq 0 \mod 2\pi$,

$$\sum_{p\geq 0} \frac{1}{m_p} t^p \quad \text{and} \quad \sum_{p\geq 0} m_p t^p$$

belong to $\mathbb{C}\{t\}$ and each of them can be extended to an infinite sector of bisecting direction θ with $q-$exponential growth of order $1/s$.

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Theorem

The sequence $([p]_{1/q}!)_{p\geq 0}$ preserves $q-$Gevrey asymptotic expansions.

Proof based on the previous characterization.

Open problems

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In [4], the authors give answer to the previous conjecture, among other related questions.

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Theorem [4, Theorem 7]

The set of sequences preserving $q-$Gevrey asymptotic expansions is strictly contained in the set of sequences preserving summability.

Further recent results

In [4], the authors describe topological results on the property of preserving summability, and on the property of preserving q-Gevrey asymptotic expansions.

Let $\tilde{m} = (\tilde{m}_p)_{p\geq 0}$ preserving summability. If $m = (m_p)_{p\geq 0}$ is such that $m_0 = 1$ and for all $s > 0$ there exist $A(s), B(s) > 0$ with

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then, m preserves summability.

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Let $\tilde{m} = (\tilde{m}_p)_{p\geq 0}$ preserving q-Gevrey asymptotic expansions. If $m = (m_p)_{p\geq 0}$ is such that $m_0 = 1$ and for all $s > 0$ there exist $A(s), B(s) > 0$ and $\alpha(s) \in \mathbb{R}$ with

$$|m_p - \tilde{m}_p| \leq \frac{1}{q^{s\frac{p(p-1)}{2}}} A(s)B(s)^p (p!)^{\alpha(s)}, \qquad p \geq 0,$$

then, m preserves q-Gevrey asymptotic expansions.

Further recent results

From the previous results, the authors prove that the sequence $m = (m_p)_{p\geq 0}$, with

$$m_p = 1 + pq^{-\frac{p(p-1)}{2}}, \quad p \geq 0$$

preserves summability but does not preserve q-Gevrey asymptotic expansions.

Thank you for your attention!