Dynamics of translation semigroups on directed metric trees

Elisabetta Mangino (joint work with Álvaro Vargas-Moreno)

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X separable infinite-dimensional Banach space

Discrete dynamics

$T : X \to X$ bounded linear operator
T hypercyclic $\Leftrightarrow \exists x \in X$ s.t.
$\overline{\{T^n x : n \in \mathbb{N}\}} = X$,

Continuous dynamics

$(T_t)_{t\geq 0}$ C_0-semigroup on X
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- Conejero-Muller-Peris 2007:
  $(T_t)_{t\geq 0}$ is hypercyclic if and only if every operator T_t is hypercyclic.
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Backward Shifts

$(v_n)_{n\in\mathbb{N}}$, $v_n > 0$, $1 \le p < \infty$.

$\ell_p(v) = \{(x_n)_{n\in\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |x_n|^p v_n < \infty\}$

$$B : \ell_p(v) \to \ell_p(v)$$
$$B((x_n))_{n\in\mathbb{N}} = (x_{n+1})_{n\in\mathbb{N}}$$

B is continuous if and only if $\sup_{n\in\mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$
$B : \ell_p(v) \to \ell_p(v)$ hypercyclic $\Leftrightarrow$ $\inf_{n\in\mathbb{N}} v_n = 0$

Translation semigroups:

$\rho : [0,+\infty[\to (0,+\infty)$ locally integrable function, $1 \le p < \infty$

$L^\rho_p = L_p([0,+\infty[, \rho dx)$

$$T_t f(\cdot) = f(\cdot + t), \qquad t \ge 0,\ f \in L^\rho_p$$

$\mathcal{T} = (T_t)_{t\ge 0}$ is a strongly continuous semigroup on L^ρ_p, called **translation semigroup**, if and only if there exist $M \ge 1$, $\omega \in \mathbb{R}$ s.t.

$$\rho(\tau) \le M e^{\omega t} \rho(\tau + t), \qquad (\tau, t > 0)$$

In this case ρ is called an admissible weight.
$\mathcal{T}$ hypercyclic on L^ρ_p $\Leftrightarrow$ $\liminf_{x\to\infty} \rho(x) = 0$.

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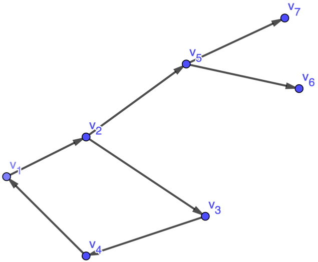
Directed trees

Martínez-Avendaño, Grosse-Erdmann, Papathanasiou, Lopez-Martínez

Let $G = (V, E)$ be a locally finite *directed tree* without cycles:

- V countable set of vertices
- $E \subset V \times V \setminus \{(v, v) \mid v \in V\}$ edges
- it can't happen that there exist $v_1, \ldots, v_k \in V$ such that (v_1, v_2), (v_2, v_3), ... (v_{k-1}, v_k), $(v_k, v_1) \in E$
- for each vertex v there is at most another vertex $w \in V$ such that w is the parent of v and v is the child of w, namely $(w, v) \in E$.
- given an edge $e = (v, w)$, v is said to be the tail of e and w the head of e
- every vertex has only a finite number of children. A vertex without children is called a *leaf*.
- A directed tree has at most one vertex without a parent, called the root of the tree and denoted root. The parent of a vertex $v \neq$ root is denoted by par(v).

$$Chi(v) = \{u \in V \mid (v, u) \in E\} \qquad Chi^n(v) = \bigcup_{u \in Chi^{n-1}(v)} Chi(u)$$



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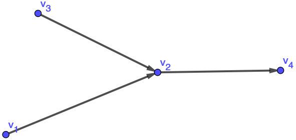

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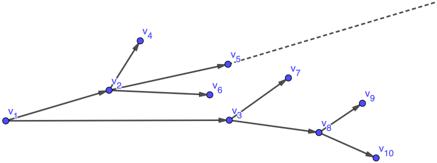
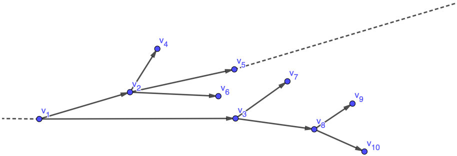




$\mathrm{par}(v_5) = v_2$ $\quad Chi(v_2) = \{v_3, v_4, v_5\}$ $\quad Chi^3(v_1) = \{v_9, v_{10}\}$

v_1 is the tail of the edge (v_1, v_2), v_2 is the head of the edge (v_1, v_2).

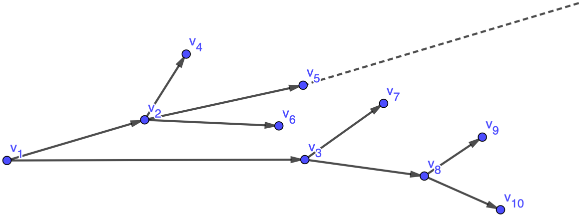
The backward shift on a tree

Let $\mu \in \mathbb{K}^V$ be a family of non-zero numbers, $1 \leq p < \infty$.

$$\ell^p(V,\mu) = \{f \in \mathbb{K}^V \mid \sum_{v \in V} |f(v)\mu_v|^p < \infty\}$$

$$(Bf)(v) = \sum_{u \in Chi(v)} f(u), \qquad v \in V$$

Example.



$Bf(v_2) = f(v_4) + f(v_5) + f(v_6)$, $Bf(v_1) = f(v_2) + f(v_3)$, $(Bf)(v_4) = 0$

Boundedness and hypercyclicity of the backward shift on directed trees

K.-G. Grosse Erdmann, D. Papathanasiou

- B is bounded on $\ell^1(V,\mu) \Leftrightarrow \sup_{v\in V\setminus \text{root}} \left|\frac{\mu_{\text{par}(v)}}{\mu_v}\right| < \infty.$
- If $1 < p < \infty$:
  B is bounded on $\ell^p(V,\mu) \Leftrightarrow \sup_{v\in V} |\mu_v|^{p^*} \sum_{u\in Chi(v)} |\mu_u|^{-p^*} < \infty.$

If the tree is rooted and without leaves:
B is hypercyclic in $\ell^p(V,\mu)$ if and only if there is an increasing sequence $(n_k)_k$ of positive integers such that for every $v \in V$

$$\lim_{k\to\infty} \inf_{u\in \text{Chi}^{n_k}(v)} |\mu_u| = 0 \qquad p = 1$$

$$\lim_{k\to\infty} \sum_{u\in \text{Chi}^{n_k}(v)} |\mu_u|^{-p^*} = \infty \qquad 1 < p < \infty$$

Evolution Equations on networks

- Heat/Diffusion: How heat or a substance diffuses through a network of pipes or wires.
- Wave Propagation: How signals (e.g., in a communication network, or pressure waves in a system of pipes) propagate and interact at junctions.
- Transport Processes: Modeling flows (e.g., traffic flow, fluid flow, information flow) in complex networks.

M. Kramar, E. Sikolya, B. Dorn, D. Mugnolo, al.

$e_j \equiv [0,1]$, $x_j(\cdot,t)$: distribution of the material along an edge e_j at time $t \geq 0$

$$\begin{cases} \partial_t x_j(s,t) = c_j \partial_s x_j(s,t), & s \in (0,1), t \geq 0, \\ \text{transport} & \\ x_j(s,0) = f_j(s), & s \in (0,1) \\ \text{initial condition} & \\ \Phi^-_{ij} c_j x_j(1,t) = w_{ij} \sum_{k=1}^m \Phi^+_{ik} c_k x_k(0,t), & t \geq 0, \\ \text{junction condition} & \end{cases}$$

for $i = 1, \ldots, n$, and $j = 1, \ldots, m$.

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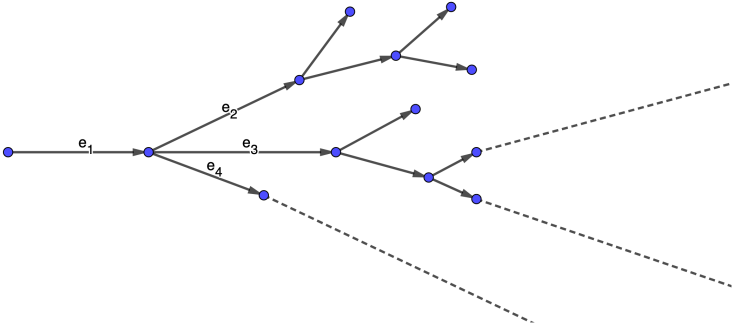
Directed metric trees

A locally finite *directed tree* without cycles $G = (V, E)$ becomes a *directed metric tree* if on every edge $e \in E$ we assign a coordinate $x_e \in [0, 1]$ which increases in the direction of the edge.
We will consider rooted directed metric trees, i.e. for which there exists an edge $e \in E$ such that for no edge $f \in E$ there exists a vertex v head of f and tail of e.



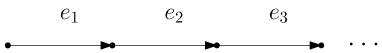
A directed metric tree $E = \{e_i \mid i \in I\}$ can be described through the matrix $\mathcal{A}$, whose entries are

$$(\mathcal{A})_{ij} = \begin{cases} 1 & \text{if there is a vertex } v \text{ such that } v \text{ is the head of } e_i \text{ and the tail of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

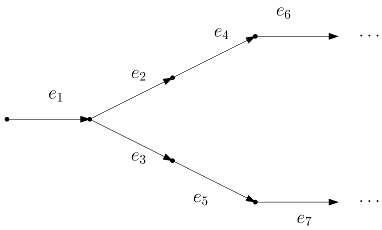
Due to the characteristics of the tree:

- $\forall j \in I: \ \text{card}(\{i \in I \mid \mathcal{A}_{ij} \neq 0\}) \leq 1$
- $\forall i \in I: \ \text{card}(\{i \in I \mid \mathcal{A}_{ij} \neq 0\}) < \aleph_0$
- $(\mathcal{A}^n)_{ij} \neq 0$ if and only if there exists a path of length n starting with edge e_i and ending with edge e_j.
- $M_n(i) = \{j \in I \mid (\mathcal{A}^n)_{ij} \neq 0\}$.

Examples

1. 

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

2. 

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Weighted L^p-spaces on directed metric trees

For every $i \in I$, let ρ_i be a strictly positive locally integrable function on $[0,1]$ and set $\rho := (\rho_i)_{i \in I}$. For every $i \in I$, consider the weighted space

$$L^p_{\rho_i}[0,1] := \{f : [0,1] \to \mathbb{R} \mid f \text{ is measurable and } |f|^p \rho_i \in L^1[0,1]\},$$

endowed with the norm

$$||f||_{p,\rho_j} = \left(\int_0^1 |f|^p \rho_i dx \right)^{\frac{1}{p}}$$

$$X^p_\rho := \ell^p \left(\oplus_{i \in I} L^p_{\rho_i}[0,1] \} \right) = \left\{ (f_i)_{i \in I} \mid f_i \in L^p_{\rho_i}[0,1] \text{ and } \sum_{i \in I} ||f_i||^p_{p,\rho_i} < \infty \right\}.$$

Remark. If the tree is of the following type:

then clearly X^p_ρ is isometric to $L^p_{\widetilde{\rho}}([0,+\infty[)$,
with $\widetilde{\rho}(s) = \rho_i(s-i)$ if $s \in [i, i+1]$, via the isometry

$$(f_i)_{i \in I} \longmapsto f$$

where $f(s) = f_i(s-i)$ if $s \in [i, i+1]$.

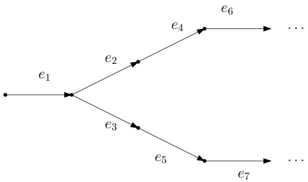
The translation semigroup on a directed metric tree

For $f \in X^p_\rho$ and $t \geq 0$:

$$(T_t f)(s) = \mathcal{A}^n f(t+s-n), \text{ for } s \in [0,1] \text{ and } n \in \mathbb{N}_0 \text{ such that } n \leq t+s < n+1,$$

The map defined in this way can intuitively be seen as a left translation with a "jump" given by the operator $\mathcal{A}$.

Example



$$(T_t f)(s) = \mathcal{A} f(t+s-1) = (f_2 + f_3, f_4, f_5, f_6, \dots)(t+s-1) \qquad 1 < t+s < 2$$

$$(T_t f)(s) = \mathcal{A}^2 f(t+s-2) = (f_4 + f_5, f_6, f_7, \dots)(t+s-2) \qquad 2 \leq t+s < 3$$

$\dots$

Proposition [M-VM]

Let ρ be a weight of the space X^p_ρ.

a. The following assertions are equivalent:
   i. The family $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on X^1_ρ;
   ii. there exist some constants $M \geq 1$ and $w \in \mathbb{R}$ such that for all $j \in I$

$$\rho_j(s) \leq Me^{wt}\rho_i(s+t-n),$$

   for all $t \geq 0$, almost every $s \in [0,1]$, $n \in \mathbb{N}_0$ such that $n \leq s+t < n+1$ and all $i \in M_n(j)$.

b. Let $1 < p < \infty$ then the following assertions are equivalent.
   i. The family $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on X^p_ρ;
   ii. there exist some constants $M \geq 1$ and $w \in \mathbb{R}$ such that for all $j \in I$

$$\left(\sum_{i\in M_n(j)} \frac{1}{\rho_i(s+t-n)^{p^*/p}} \right)^{p/p^*} \leq Me^{wt}\frac{1}{\rho_j(s)}$$

   for all $t \geq 0$, almost every $s \in [0,1]$, $n \in \mathbb{N}_0$ such that $n \leq s+t < n+1$.

Remark.

If G has a leaf, then the translation semigroup cannot be hypercyclic in any admissible space X^p_ρ.

Indeed, if (v_{i_0}, v_{j_0}) are such that v_{j_0} is a leaf of G and there exists some edge $e_{k_0} = (v_{i_0}, v_{j_0})$, then $(T_t f)_{k_0}(s) = 0$ for all $t > 1$ and $0 \le s \le 1$, for every $f \in X^p_\rho$ and therefore the semigroup cannot be hypercyclic.

Theorem [M-VM]

Let $G = (V, E)$ be a rooted directed metric tree without leafs.

a. The following assertions are equivalent:
   1. the translation semigroup is hypercyclic on X^1_ρ;
   2. the translation semigroup is weakly mixing on X^1_ρ;
   3. there exists some increasing $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ such that for all $j \in I$:

$$\lim_{k\to\infty} \inf_{s\in[0,1]} \inf_{i\in M_{n_k}(j)} \rho_i(s) = 0.$$

b. The following assertions are equivalent:
   1. the translation semigroup is hypercyclic on X^p_ρ;
   2. the translation semigroup is weakly mixing on X^p_ρ;
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$$\lim_{k\to 0} \inf_{s\in[0,1]} \left(\sum_{i\in M_{n_k}(j)} \frac{1}{\rho_i(s)^{p^*/p}} \right)^{-1} = 0.$$

Remark.

$$\lim_{k\to 0} \inf_{s\in[0,1]} \left(\left(\mathcal{A}^{n_k} \frac{1}{\rho(s)(s)^{p^*/p}} \right)_j \right)^{-1} = 0$$

$$\left(\frac{1}{\rho^{p^*/p}}\right)_i = \frac{1}{\rho_i(s)^{p^*/p}}$$

Work in progress:

- study of chaos, frequent hypercylicity,...
- spaces of continuous functions on directed metric trees with suitable "junctions" in the vertices.