Topological properties of weighted composition operators in sequence spaces

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Topic

For fixed sequences $u = (u_i)_{i\in\mathbb{N}}, \varphi = (\varphi_i)_{i\in\mathbb{N}}$, we consider the weighted composition operator $W_{u,\varphi}$ with symbols u, φ defined by $x \mapsto u(x \circ \varphi) = (u_i x_{\varphi_i})_{i\in\mathbb{N}}$. We characterize the continuity and the compactness of the operator $W_{u,\varphi}$ when it acts on the weighted Banach spaces $l^p(v)$, $1 \leq p \leq \infty$, and $c_0(v)$, with $v = (v_i)_{i\in\mathbb{N}}$ a weight sequence on $\mathbb{N}$. We extend these results to the case in which the operator $W_{u,\varphi}$ acts on Köthe echelon and co-echelon spaces, sequence (LF)-spaces of type $l_p(\mathcal{V})$ and on (PLB)-spaces of type $a_p(\mathcal{V})$, with $p \in [1, \infty] \cup \{0\}$ and $\mathcal{V}$ a system of weights on $\mathbb{N}$.

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Outline

1. Weighted composition operators on Banach spaces;
2. Weighted composition operators on (PLB)- and (LF)-spaces.

Weighted Composition Operators or Pseudo Shifts

For fixed $u = (u_i)_{i\in\mathbb{N}}, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$, we can define the weighted composition operator $W_{u,\varphi}$ acting on ω with symbols u, φ by setting

$$W_{u,\varphi}(x) := u(x \circ \varphi) = (u_i x_{\varphi_i})_{i\in\mathbb{N}}, \quad x = (x_i)_{i\in\mathbb{N}} \in \omega.$$

This operator is obtained by composition of two well-known operators: the multiplication operator M_u and the composition operator C_φ. In fact, when φ is the identity map on $\mathbb{N}$, $W_{u,\varphi}$ becomes a multiplication operator which is defined pointwise by $M_u(x) := ux = (u_i x_i)_{i\in\mathbb{N}}$. If $u_i = 1$ for all $i \in \mathbb{N}$, then $W_{u,\varphi}$ becomes a composition operator defined as $C_\varphi(x) := x \circ \varphi = (x_{\varphi_i})_{i\in\mathbb{N}}$. Clearly, $W_{u,\varphi} \in \mathcal{L}(\omega)$ for every pair $u, \varphi \in \omega$.

We denote by e_n, for $n \in \mathbb{N}$, the element $(\delta_{n,i})_{i\in\mathbb{N}}$ of ω.

Sequence $l^p(v)$ spaces

Given a weight v, i.e., a positive sequence $v = (v_i)_{i\in\mathbb{N}}$ on $\mathbb{N}$ and $1 \leq p \leq \infty$, we define as usual

$$l^p(v) := \{x = (x_i)_{i\in\mathbb{N}} \in \omega \colon \|x\|_{p,v} := \|(x_i v_i)_{i\in\mathbb{N}}\|_p < \infty\},$$

where $\|\cdot\|_p$ denotes the usual l^p norm. For $p = 0$, we set

$$c_0(v) := \left\{x = (x_i)_{i\in\mathbb{N}} \in \omega \colon \lim_{i\to\infty} v_i x_i = 0\right\}.$$

Clearly, $(l^p(v), \|\cdot\|_{p,v})$, $1 \leq p \leq \infty$, are Banach spaces, and $c_0(v)$ is a Banach space with the norm of $l^\infty(v)$.

Continuity of $W_{u,\varphi}$

Theorem

Let $u = (u_i)_{i\in\mathbb{N}}, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$, let v, w be two weights on $\mathbb{N}$ and $1 \leq p < \infty$. The weighted composition operator $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$ if, and only if, there exists $M > 0$ such that

$$\frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \leq M, \quad \forall n \in \mathbb{N},$$

where the sum is defined equal to 0 if $\varphi^{-1}(n) = \emptyset$ for some $n \in \mathbb{N}$.

For $p = 2$ the result was given in L.H. Khoi, D. Luan, *Weighted composition operators on weighted sequence spaces*, Contemp. Math. **645** (2015), 199–215.

The proof

Proof

If the inequality is satisfied, then for every $x \in l^p(v)$ we have

$$\|W_{u,\varphi}(x)\|_{p,w}^p = \sum_{j\in\mathbb{N}} |u_j|^p |x_{\varphi_j}|^p w_j^p = \sum_{n\in\mathbb{N}} \sum_{j\in\varphi^{-1}(n)} |u_j|^p |x_n|^p w_j^p$$

$$= \sum_{n\in\mathbb{N}} |x_n|^p \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \leq M \sum_{n\in\mathbb{N}} |x_n|^p v_n^p = M\|x\|_{p,v}^p.$$

This means that $W_{u,\varphi} \in \mathcal{L}(l^p(v), l^p(w))$. Conversely, there exists $M > 0$ such that $\|W_{u,\varphi}(x)\|_{p,w}^p \leq M\|x\|_{p,v}^p$ for every $x \in l^p(v)$. Fix $n \in \mathbb{N}$ such that $\varphi^{-1}(n) \neq \emptyset$. Observe that $W_{u,\varphi}(e_n) = (u_j(e_n)_{\varphi_j})_{j\in\mathbb{N}} = (u_j\delta_{n,\varphi_j})_{j\in\mathbb{N}}$. Therefore, if $x = e_n$, we get that

$$\sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p = \|W_{u,\varphi}(e_n)\|_{p,w}^p \leq M\|e_n\|_{p,v}^p = v_n^p. \quad \square$$

Continuity of $W_{u,\varphi}$

Remark

For $p = \infty$ the operator $W_{u,\varphi}$ belongs to $\mathcal{L}(l^\infty(v), l^\infty(w))$ if, and only if, $\sup_{n\in\mathbb{N}} \frac{\|W_{u,\varphi}(e_n)\|_{\infty,w}}{\|e_n\|_{\infty,v}} < \infty$[1]. This is equivalent to the existence of $M > 0$ such that

$$\sup_{j\in\varphi^{-1}(n)} |u_j|w_j \leq M v_n, \quad \forall n \in \mathbb{N}.$$

Remark

If we assume that $\varphi \colon \mathbb{N} \to \mathbb{N}$ is a proper map, then the operator $W_{u,\varphi}$ belongs to $\mathcal{L}(c_0(v), c_0(w))$ if, and only if, $W_{u,\varphi}$ belongs to $\mathcal{L}(l^\infty(v), l^\infty(w))$.

[1] C. Carpintero, J.C. Ramos-Fernández, J.E. Sanabria, *Weighted composition operators between two different weighted sequence spaces*, Advances in Pure and Applied Mathematics **13**(2) (2022), 29–42.

Compactness of $W_{u,\varphi}$

Theorem

Let $u = (u_i)_{i\in\mathbb{N}}, \varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$, let v, w be two weights on $\mathbb{N}$ and $1 \leq p < \infty$. Then $W_{u,\varphi} \in \mathcal{K}(l^p(v), l^p(w))$ if, and only if,

$$\left(\frac{1}{v_n^p} \sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p \right)_{n\in\mathbb{N}} \in c_0.$$

For $p = 2$ the result was given in L.H. Khoi, D. Luan, *Weighted composition operators on weighted sequence spaces*, Contemp. Math. **645** (2015), 199–215.

The proof

An idea

Case: $1 < p < \infty$. Suppose that the operator $W_{u,\varphi}$ is compact. As $\left\{\frac{e_n}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$ is a bounded subset of $l^p(v)$, it then follows that the set $\left\{\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$ is relatively compact in $l^p(w)$. On the other hand, the sequence $\left(\frac{e_n}{\|e_n\|_{p,v}}\right)_{n\in\mathbb{N}}$ weakly converges to 0 in $l^p(v)$, thus the set $\left\{\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} : n \in \mathbb{N}\right\}$ is relatively weakly compact in $l^p(w)$. Thus, $\frac{W_{u,\varphi}(e_n)}{\|e_n\|_{p,v}} \to 0$ in $l^p(w)$.

Conversely, suppose that $\left(\frac{1}{v_n^p}\sum_{j\in\varphi^{-1}(n)} |u_j|^p w_j^p\right)_{n\in\mathbb{N}} \in c_0$. We fix a bounded sequence $(x_i)_{i\in\mathbb{N}}$ of $l^p(v)$ and since $l^p(v)$ is a reflexive Banach space, there exists a subsequence of $(x_i)_{i\in\mathbb{N}}$, denoted again by $(x_i)_{i\in\mathbb{N}}$ for the sake of simplicity, that weakly converges in $l^p(v)$ to some $x \in l^p(v)$. It can be proved that $W_{u,\varphi}(x_i) \to W_{u,\varphi}(x)$ in $l^p(w)$.

The proof

An idea

Case: $p = 1$. Suppose that $W_{u,\varphi}$ is compact. Due to a Lemma, the dual operator $W'_{u,\varphi} \in \mathcal{L}\left(l^\infty\left(\frac{1}{w}\right), l^\infty\left(\frac{1}{v}\right)\right)$ maps $c_0\left(\frac{1}{w}\right)$ into $c_0\left(\frac{1}{v}\right)$ and $T := W'_{u,\varphi}|_{c_0\left(\frac{1}{w}\right)} \in \mathcal{L}\left(c_0\left(\frac{1}{w}\right), c_0\left(\frac{1}{v}\right)\right)$. Therefore, $W_{u,\varphi} = T' \in \mathcal{L}(l^1(v), l^1(w))$ is also $\sigma\left(l^1(v), c_0(\frac{1}{v})\right) - \sigma\left(l^1(w), c_0(\frac{1}{w})\right)$ continuous, i.e., w*-w* continuous. Arguing as before, we get that the set $\left\{W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right) : n \in \mathbb{N}\right\}$ is relatively weakly* compact and relatively compact in $l^1(w)$, thereby implying that $W_{u,\varphi}\left(\frac{e_n}{\|e_n\|_{1,v}}\right) \to 0$ in $l^1(w)$.

On the other way, if $\left(\frac{1}{v_n}\sum_{j\in\varphi^{-1}(n)} |u_j| w_j\right)_{n\in\mathbb{N}} \in c_0$ and $(x_i)_{i\in\mathbb{N}}$ is a bounded sequence of $l^1(v)$, since $l^1(v) \hookrightarrow \omega$, there exists a subsequence of $(x_i)_{i\in\mathbb{N}}$ convergent in ω to some $x \in \omega$. It can be shown that $x \in l^1(v)$ and $W_{u,\varphi}(x_i) \to W_{u,\varphi}(x)$ in $l^1(w)$. $\square$

Remark

For $p=\infty$ the operator $W_{u,\varphi}$ belongs to $\mathcal{K}(l^\infty(v), l^\infty(w))$ if, and only if, we have $\lim_{n\to\infty} \frac{\|W_{u,\varphi}(e_n)\|_{\infty,w}}{\|e_n\|_{\infty,v}} = 0$ [2]. This is equivalent to require that

$$\lim_{n\to\infty} \frac{\sup_{j\in\varphi^{-1}(n)} |u_j| w_j}{v_n} = 0.$$

If $p=0$ and $\varphi\colon \mathbb{N}\to\mathbb{N}$ is a proper map, then the operator $W_{u,\varphi}$ belongs to $\mathcal{K}(c_0(v), c_0(w))$ if, and only if, it belongs to $\mathcal{K}(l^\infty(v), l^\infty(w))$.

[2] C. Carpintero, J.C. Ramos-Fernández, J.E. Sanabria, *Weighted composition operators between two different weighted sequence spaces*, Advances in Pure and Applied Mathematics **13**(2) (2022), 29–42.

(LF)- and (PLB)-spaces

Definition

A lcHs E is called an (LF)-space if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of Fréchet spaces with $E_n \hookrightarrow E_{n+1}$ continuously such that $E = \bigcup_{n\in\mathbb{N}} E_n$ and the topology of E coincides with the finest locally convex topology for which each inclusion $E_n \hookrightarrow E$ is continuous. In such a case, we simply write $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$. The space $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ is called an (LB)-space if E_n is a Banach space for all $n \in \mathbb{N}$.
A lcHs E is called a (PLB)-space if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of (LB)-spaces with $E_{n+1} \hookrightarrow E_n$ continuously, for $n \in \mathbb{N}$, such that $E = \bigcap_{n\in\mathbb{N}} E_n$ and the topology of E is the coarsest locally convex topology for which each inclusion $E \hookrightarrow E_n$ is continuous. In such a case, we simply write $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$.

Properties of (LF)-spaces

An (LF)-space $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$ is called *regular* (*compactly retractive*, resp.) if every bounded (compact, resp.) subset B of E is contained and bounded (compact, resp.) in E_n for some $n \in \mathbb{N}$.

- Every complete (LF)-space is always regular.
- Characterization of regularity of (LF)-space.
- Continuity of linear operators between (LF)-spaces due to Grothendieck.
- Let $E = \operatorname{ind}_{m\in\mathbb{N}} E_m$ and $F = \operatorname{ind}_{n\in\mathbb{N}} F_n$ be two (LF)-spaces. Let $T\colon E \to F$ be a linear operator and assume that F is compactly retractive. The operator T is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T\colon E_m \to F_n$ is compact.

Properties of (PLB)-spaces

- A (PLB)-space $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ is complete whenever E_n is a complete (LB)-space for an infinite number of indices n.
- Let $E = \operatorname{proj}_{n\in\mathbb{N}} E_n$ be a (PLB)-space such that the continuous inclusion $E \hookrightarrow E_n$ has dense range for all $n \in \mathbb{N}$. Let $F = \operatorname{proj}_{k\in\mathbb{N}} F_k$ be a (PLB)-space such that F_k is a complete (LB)-space for all $k \in \mathbb{N}$. Let $T\colon E \to F$ be a linear operator. Then the following assertions hold true:
    1. The operator T is continuous if, and only if, for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the operator T admits a unique linear continuous extension $T_k^n\colon E_n \to F_k$.
    2. The operator T is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator T admits a unique linear extension $T_k^n\colon E_n \to F_k$ which is compact.

Systems of weights

For all $n \in \mathbb{N}$, let $V_n = (v_{n,k})_{k\in\mathbb{N}}$ be a countable family of (strictly) positive sequences, called *weights*, on $\mathbb{N}$. We denote by $\mathcal{V}$ the sequence $(V_n)_{n\in\mathbb{N}}$ and we assume that the following two conditions are satisfied:

1. $v_{n,k}(i) \le v_{n,k+1}(i)$ for all $n, k \in \mathbb{N}$ and $i \in \mathbb{N}$;
2. $v_{n,k}(i) \ge v_{n+1,k}(i)$ for all $n, k \in \mathbb{N}$ and $i \in \mathbb{N}$.

The family $\mathcal{V}$ is called a *system of weights* on $\mathbb{N}$.

Sequence (LF)-spaces

Since $l^p(v_{n,k+1})$ is continuously included into $l^p(v_{n,k})$, the sequence $\{l^p(v_{n,k})\}_{k\in\mathbb{N}}$ of Banach spaces forms a projective spectrum. Hence, for all $n \in \mathbb{N}$ and $1 \le p \le \infty$, we can consider the *Köthe echelon Fréchet spaces*

$$\lambda_p(V_n) := \bigcap_{k\in\mathbb{N}} l^p(v_{n,k}) \text{ and } \lambda_0(V_n) := \bigcap_{k\in\mathbb{N}} c_0(v_{n,k}).$$

The sequence $\{\lambda_p(V_n)\}_{n\in\mathbb{N}}$ of Fréchet spaces forms an inductive spectrum. Thus, the spaces

$$l_p(\mathcal{V}) := \bigcup_{n\in\mathbb{N}} \lambda_p(V_n) \ \ (1 \le p \le \infty) \text{ and } l_0(\mathcal{V}) := \bigcup_{n\in\mathbb{N}} \lambda_0(V_n)$$

endowed with the inductive topologies, i.e., $l_p(\mathcal{V}) = \operatorname{ind}_{n\in\mathbb{N}} \lambda_p(V_n)$ ($l_0(\mathcal{V}) = \operatorname{ind}_{n\in\mathbb{N}} \lambda_0(V_n)$, resp.) are (LF)-spaces.

Sequence (PLB)-spaces

Both the sequences $\{l^p(v_{n,k})\}_{n\in\mathbb{N}}$ and $\{c_0(v_{n,k})\}_{n\in\mathbb{N}}$ of Banach spaces form an inductive spectrum. Hence, we can consider the *Köthe co-echelon spaces*

$$a_p(V^k) := \bigcup_{n\in\mathbb{N}} l^p(v_{n,k}) \quad (1 \leq p \leq \infty) \text{ and } a_0(V^k) := \bigcup_{n\in\mathbb{N}} c_0(v_{n,k}),$$

which are (LB)-spaces when they are endowed with the inductive topologies, i.e., $a_p(V^k) = \operatorname{ind}_{n\in\mathbb{N}} l^p(v_{n,k})$
($a_0(V^k) = \operatorname{ind}_{n\in\mathbb{N}} c_0(v_{n,k})$, resp.). The sequence $\{a_p(V^k)\}_{k\in\mathbb{N}}$ of (LB)-spaces forms a projective spectrum. Hence, the spaces

$$a_p(\mathcal{V}) := \bigcap_{k\in\mathbb{N}} a_p(V^k) \quad (1 \leq p \leq \infty) \text{ and } a_0(\mathcal{V}) := \bigcap_{k\in\mathbb{N}} a_0(V^k),$$

endowed with the projective topologies, i.e.,
$a_p(\mathcal{V}) = \operatorname{proj}_{k\in\mathbb{N}} a_p(V^k)$ and $a_0(\mathcal{V}) = \operatorname{proj}_{k\in\mathbb{N}} a_0(V^k)$ are (PLB)-spaces.

Continuity of $W_{u,\varphi}$ between sequence (LF)-spaces

Theorem

Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p < \infty$, the following properties are equivalent:

1. $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is well-defined;
2. $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous;
3. For all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$, $M > 0$ for which

$$\frac{1}{v_{m,l}^p(i)} \sum_{j\in\varphi^{-1}(i)} |u_j|^p w_{n,k}^p(j) \leq M, \quad \forall i \in \mathbb{N},$$

   where the sum is defined equal to 0 if $\varphi^{-1}(i) = \emptyset$ for some $i \in \mathbb{N}$.

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Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. If $p = \infty$, the following properties are equivalent:

1. $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is well-defined;
2. $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is continuous;
3. For all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$, $M > 0$ for which

$$\sup_{j\in\varphi^{-1}(i)} |u_j| w_{n,k}(j) \leq M v_{m,l}(i), \quad \forall i \in \mathbb{N}.$$

If $p = 0$ and $\varphi\colon \mathbb{N} \to \mathbb{N}$ is a proper map, then $W_{u,\varphi}\colon l_0(\mathcal{V}) \to l_0(\mathcal{W})$ is continuous if, and only if, $W_{u,\varphi}\colon l_\infty(\mathcal{V}) \to l_\infty(\mathcal{W})$ is continuous.

Continuity of $W_{u,\varphi}$ between sequence (PLB)-spaces

Theorem

Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi, u \in \omega$.

1. If $1 \leq p < \infty$, then $W_{u,\varphi} \colon a_p(\mathcal{V}) \to a_p(\mathcal{W})$ is continuous if, and only if, $\forall k \in \mathbb{N}\ \exists l \in \mathbb{N}$ st $\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}$, $M > 0$:
$$\frac{1}{v_{m,l}^p(i)} \sum_{j \in \varphi^{-1}(i)} |u_j|^p w_{n,k}^p(j) \leq M, \quad \forall i \in \mathbb{N}.$$
2. If $p = \infty$, then $W_{u,\varphi} \colon a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ is continuous if, and only if, $\forall k \in \mathbb{N}\ \exists l \in \mathbb{N}$ st $\forall m \in \mathbb{N}\ \exists n \in \mathbb{N}$, $M > 0$:
$$\sup_{j \in \varphi^{-1}(i)} |u_j| w_{n,k}(j) \leq M v_{m,l}(i), \quad \forall i \in \mathbb{N}.$$
3. If $\varphi \colon \mathbb{N} \to \mathbb{N}$ is a proper map and $W^k = (w_{n,k})_{n \in \mathbb{N}}$ is regularly decreasing $\forall k \in \mathbb{N}$, then $W_{u,\varphi} \in \mathcal{L}(a_0(\mathcal{V}), a_0(\mathcal{W})$ if, and only if, $W_{u,\varphi} \in \mathcal{L}(a_\infty(\mathcal{V}), a_\infty(\mathcal{W}))$.

Compactness of $W_{u,\varphi}$ between sequence (LF)-spaces

Theorem

Let $\mathcal{V}, \mathcal{W}$ be two system of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. The following assertions hold true:

1. If $1 \leq p < \infty$ and $l_p(\mathcal{W})$ is compactly retractive, then $W_{u,\varphi}\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\lim_{i\to\infty} \frac{1}{v_{m,l}^p(i)} \sum_{j\in\varphi^{-1}(i)} |u_j|^p w_{n,k}^p(j) = 0.$$

Compactness of $W_{u,\varphi}$ between sequence (LF)-spaces

Theorem

1. If $p=\infty$, $l_\infty(\mathcal{W})$ is compactly retractive and the space $\lambda_\infty(V_m)$ is dense in $l^\infty(v_{m,l})$ for all $m,l\in\mathbb{N}$, then $W_{u,\varphi}\colon l_\infty(\mathcal{V})\to l_\infty(\mathcal{W})$ is compact if, and only if, there exists $n\in\mathbb{N}$ such that for all $m\in\mathbb{N}$ there exists $l\in\mathbb{N}$ such that for all $k\in\mathbb{N}$
$$\lim_{i\to\infty}\frac{\sup_{j\in\varphi^{-1}(i)}|u_j|w_{n,k}(j)}{v_{m,l}(i)}=0.$$
2. If $p=0$, $l_0(\mathcal{W})$ is compactly retractive and $\varphi\colon\mathbb{N}\to\mathbb{N}$ is a proper map, then $W_{u,\varphi}\colon l_0(\mathcal{V})\to l_0(\mathcal{W})$ is compact if, and only if, $W_{u,\varphi}\colon l_\infty(\mathcal{V})\to l_\infty(\mathcal{W})$ is compact.

Theorem

Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. The following assertions hold true:

1. If $1 \leq p < \infty$, then $W_{u,\varphi}\colon a_p(\mathcal{V}) \to a_p(\mathcal{W})$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$

$$\lim_{i\to\infty} \frac{1}{v_{m,l}^p(i)} \sum_{j\in\varphi^{-1}(i)} |u_j|^p w_{n,k}^p(j) = 0.$$

Compactness of $W_{u,\varphi}$ between sequence (PLB)-spaces

Theorem

Let $\mathcal{V}, \mathcal{W}$ be two systems of weights on $\mathbb{N}$ and $\varphi = (\varphi_i)_{i\in\mathbb{N}}, u = (u_i)_{i\in\mathbb{N}} \in \omega$. The following assertions hold true:

1. If $p = \infty$, $a_\infty(\mathcal{V})$ is dense in $a_\infty(V^l)$ for all $l \in \mathbb{N}$ and the sequence $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$, then $W_{u,\varphi} \colon a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$
$$\lim_{i\to\infty} \frac{\sup_{j\in\varphi^{-1}(i)} |u_j| w_{n,k}(j)}{v_{m,l}(i)} = 0.$$
2. If $p = 0$, $\varphi \colon \mathbb{N} \to \mathbb{N}$ is a proper map and the sequence $W^k = (w_{n,k})_{n\in\mathbb{N}}$ is regularly decreasing for all $k \in \mathbb{N}$, then $W_{u,\varphi} \colon a_0(\mathcal{V}) \to a_0(\mathcal{W})$ is compact if, and only if, $W_{u,\varphi} \colon a_\infty(\mathcal{V}) \to a_\infty(\mathcal{W})$ is compact.

Main references

1. A. A. Albanese, C. Mele, *On composition operators between weighted (LF)- and (PLB)-spaces of continuous functions*, Math. Nachr. **296** (12) (2023), 1–16.
2. A. A. Albanese, C. Mele, *Topological properties of weighted composition operators in sequence spaces*, Results Math **78** (2023), Article number 210.

THANK FOR YOUR ATTENTION!

HAPPY BIRTHDAY PEPE!!

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