

w^* - w -points of continuity of the dual unit ball of a Banach space

S. Daptari¹, V. Montesinos², T.S.S.R.K. Rao¹

¹Shiv Nadar University ²Universitat Politècnica de València and Academia de Ciencias

Functional Analysis Meeting on the occasion of the 70th
birthday of José Bonet





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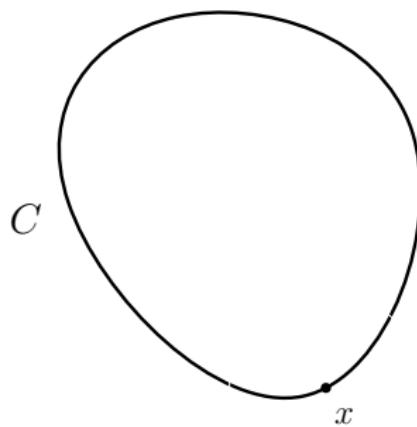
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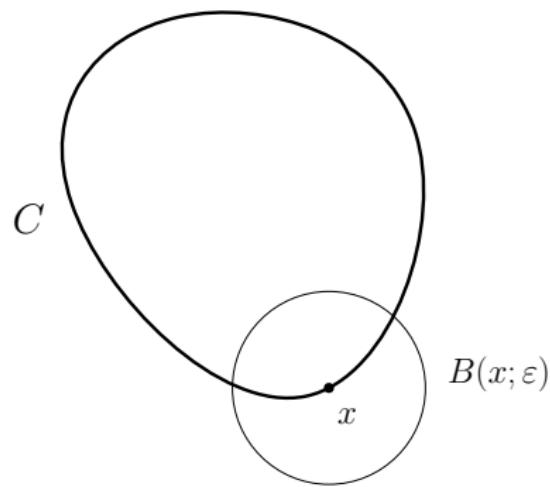
Outline

- 1 Why points of w^*-w -continuity?
- 2 Points of w^*-w -continuity and Hahn–Banach extensions
- 3 Stability
- 4 M-embedded spaces

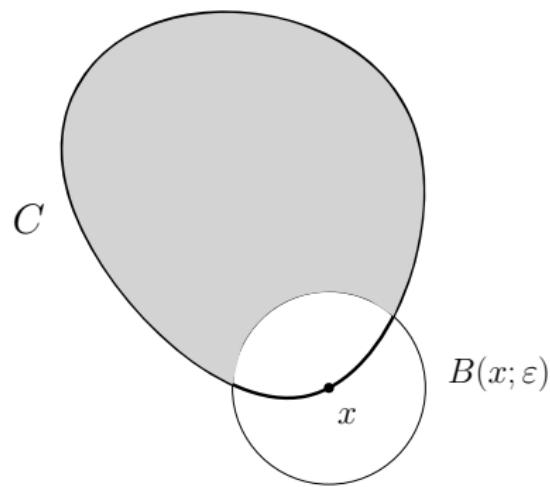
Denting points



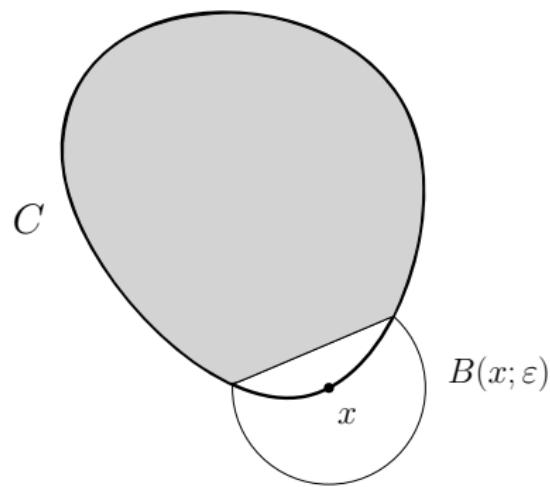
Denting points



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Theorem (Lin–Lin–Troyanski'88)

Denting point \Leftrightarrow extremal + w - $\|\cdot\|$ -continuity.

Denting points

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Theorem (Rieffel'67)

C convex closed.

C RNP $\Leftrightarrow D = \overline{\text{co}}(\text{denting points of } D), \forall D \subset C$ convex closed.

Locating the points of w^*-w -continuity

X nonreflexive Banach space

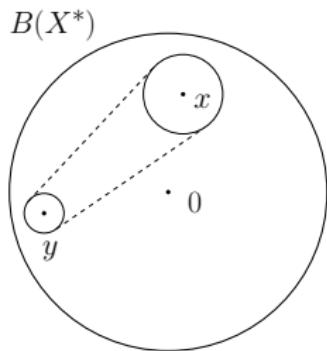
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$$\text{id} : (B(X^*), w^*) \rightarrow (B(X^*), w)$$

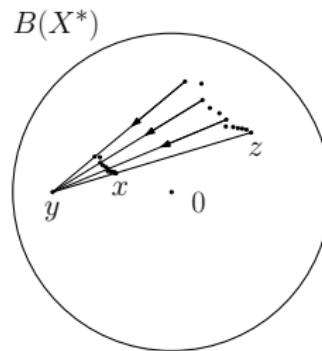
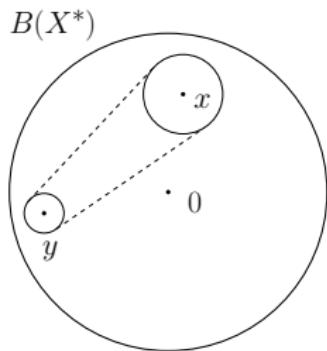
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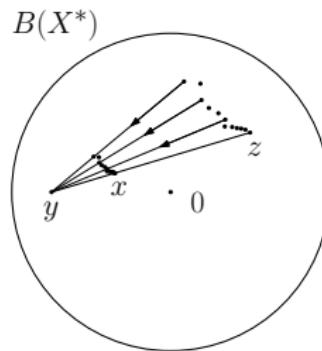
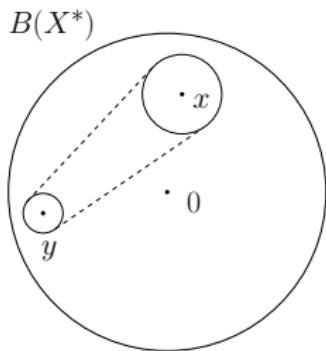
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Thus, the points of w^*-w -continuity are **on the sphere**.

w^*-w -continuity, Hahn–Banach extensions, and renormings

Lemma (G. Godefroy'81)

$x^* \in S(X^*)$. TFAE:

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- x^* has **unique** Hahn–Banach extension to X^{**} .

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Theorem (Phelps, Godefroy)

TFAE

- $id : (B(X^*), w^*) \rightarrow (B(X^*), w)$ continuous *at all* $x^* \in S(X^*)$.

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Exercise: Prove that then X is Asplund

Raja's renorming theorem

Theorem (M. Raja'02)

All $x^* \in S(X^*)$ of w^*-w -continuity.

Raja's renorming theorem

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All $x^ \in S(X^*)$ of w^*-w -continuity. $\implies \exists$ equivalent $\|\cdot\|$ on X^* dual and LUR*

Raja's renorming theorem

Theorem (M. Raja'02)

All $x^ \in S(X^*)$ of w^* - w -continuity. $\implies \exists$ equivalent $\|\cdot\|$ on X^* dual and LUR (and so $\|\cdot\|$ on X is Fréchet, and X is Asplund).*

Stability when going up

Theorem

X^{**}/X reflexive. If $x^* \in S(X^*)$ of continuity of $\text{id} : (B(X^*), w^*) \rightarrow (B(X^*), w)$, then of continuity in **all odd higher duals**.

Stability when going up

Theorem

X^{**}/X reflexive. If $x^* \in S(X^*)$ of continuity of $\text{id} : (B(X^*), w^*) \rightarrow (B(X^*), w)$, then of continuity in **all odd** higher duals. You can add “and x^* **extreme point**”.

No stability when going up

Theorem

X^{**}/X nonreflexive separable. Then \exists renorming Z of X and $x^* \in S(Z^*)$ of continuity for $\text{id} : (B(Z^*), w^*) \rightarrow (B(Z^*), w)$ and not for $\text{id} : (B(Z^{***}), w^*) \rightarrow (B(Z^{***}), w)$

No stability when going down

Theorem

$X = L^1[0, 1]$. Then $id : (B(X^{(IV)}), w^*) \rightarrow (B(X^{IV}), w)$ has an extreme point of continuity, but $id : (B(X^{**}), w^*) \rightarrow (B(X^{**}), w)$ is nowhere continuous.

M-embedded spaces

M-embedded space X means M-ideal in X^{**} .

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X non-reflexive M-embedded in X^{**} . Then

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Example: c_0 in ℓ^∞ .

Theorem

X non-reflexive M-embedded in X^{**} . Then

- $id : (B(X^{***}), w^*) \rightarrow (B(X^{***}), w)$ continuous precisely at $S(X^*)$.

M-embedded spaces

M-embedded space X means M-ideal in X^{**} .

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Theorem

X non-reflexive M-embedded in X^{**} . Then

- $id : (B(X^{***}), w^*) \rightarrow (B(X^{***}), w)$ continuous precisely at $S(X^*)$.
- There is no point of continuity of $id : (B(X^{**}), w^*) \rightarrow (B(X^{**}), w)$.

M-embedded spaces

M-embedded space X means M-ideal in X^{**} .

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Theorem

X non-reflexive M-embedded in X^{**} . Then

- $id : (B(X^{***}), w^*) \rightarrow (B(X^{***}), w)$ continuous precisely at $S(X^*)$.
- There is no point of continuity of
 $id : (B(X^{**}), w^*) \rightarrow (B(X^{**}), w)$.

Corollary

The set of points of continuity of

$id : B(\mathcal{L}(\ell^p)^*, w^*) \rightarrow (B(\mathcal{L}(\ell^p)^*, w)$ is precisely $S(\mathcal{K}(\ell^p)^*)$.



References I

-  M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler.
Banach Space Theory: the Basis for Linear and Non-Linear Analysis
Springer-Verlag, New York, 2011.
-  S. Daptari, V. Montesinos, and T.S.S.R.K. Rao
A study of points of w^ - w -continuity in the dual unit ball of a Banach space.*
To appear in RACSAM.



"IT'S MOST EXTRAORDINARY ---
AND HE IS ONLY 70."

Banach Spaces and Related Topics

January 30th - February 3rd, 2023 KIAS(Korea Institute of Advanced Study) Lecture Room on the 1st floor of 8th Building

THE 18th IISU SCHOOL OF MATHEMATICS

TAEKWANG ILJU KOREA KIPS PMI POSTECH KIAS

From: Hahn (1941), Banach 1948

X Banach $\cup_{x \in X} \exists \subset \text{closed ball}$ if x is compact
 $\rightarrow \exists$ Banach space st. $X \subset Y'$

$\forall x, y \in X^* \mid \exists \cup_x \subset \text{closed ball}$
 $\forall x, y \in X^* \subset Y \subset (x, y)$

From that Y closed in $(X, \|\cdot\|)$ $Y' \subset X$

B Euclidean unit ball in C^0 $N = \mathbb{D}$
 S sphere $N = \mathbb{T}$

S = rotation invariant measure on S $\sigma(S) = 1$
 $\text{Every } (0,1) \text{ normalized Lebesgue measure on } \mathbb{T}$

$B \supset A(B) \supset \{f \in C(B), \|f\|_B \leq 1\}$
 $\cup_{n \in \mathbb{N}} \subset \text{compact}$ (\subset = compact w.r.t.)

$\|f\|_B = \sup_{z \in B} |f(z)|$ $N = \mathbb{H}$

$A(S)$

$L^\infty(\mathbb{B}) = \{f \in H^\infty(\mathbb{B}) \mid \exists \cup_{n \in \mathbb{N}} \subset \text{closed ball}$

$H^\infty(\mathbb{B}) \not\supset G^\infty(\mathbb{B}) \supset \{f \in L^\infty(\mathbb{B}) \mid \exists \cup_{n \in \mathbb{N}} \subset \text{closed ball}$

Ando M. $H^\infty(\mathbb{D})$ has a unique predual. $(L^\infty(\mathbb{D}))^*$

Problem (Aren): When $f \in H^\infty(\mathbb{B})$ are norm attained

on $G^\infty(\mathbb{B})$ $\in \text{NA}(G^\infty(\mathbb{B}))$?

Te (Fatou) $\forall f \in H^\infty(\mathbb{B}) = \{f^*(z) = \lim_{n \rightarrow \infty} f(z_n) \mid z_n \in \mathbb{B}\}$
 $H^\infty(\mathbb{B}) \rightarrow L^\infty(\mathbb{B}), f \mapsto f^*$ is norm preserving

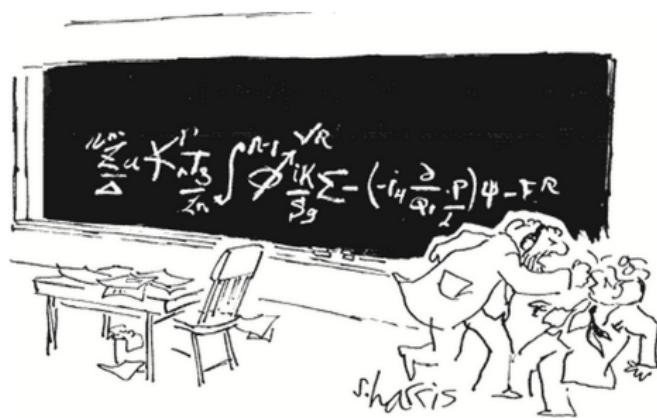
II (Fisher 1957) $\{f \in H^\infty(\mathbb{B}), \|f\|_B = 1\}$
 $\{f \in \text{NA}(G^\infty(\mathbb{B})) \mid \exists \cup_{n \in \mathbb{N}} \subset \text{closed ball} \}$ has positive measure

Example: (1) $g(z) = e^z \in \text{NA}(G^\infty(\mathbb{D}))$
 $\cup_{n \in \mathbb{N}} \subset \text{closed ball}$ ($\cup_{n \in \mathbb{N}} \subset \text{closed ball}$)

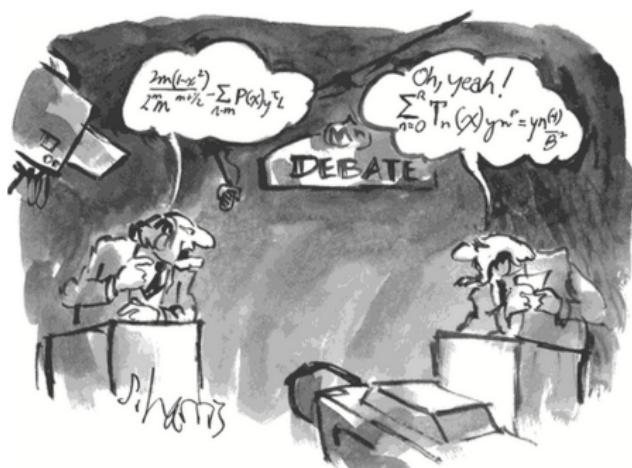
(2) $g(z) = \frac{1+z}{2} \in H^\infty(\mathbb{B})$ $\|g\|_B = 1$
 $\forall z \in \mathbb{T} \setminus \{g(z) = 1\} = \mathbb{H}$

That is, $\exists g \in A(\mathbb{D})$ s.t. $\|g(z)\| = 1 \forall z \in \mathbb{T}$
 $|g(z)| < 1 \forall z \in \mathbb{T}$

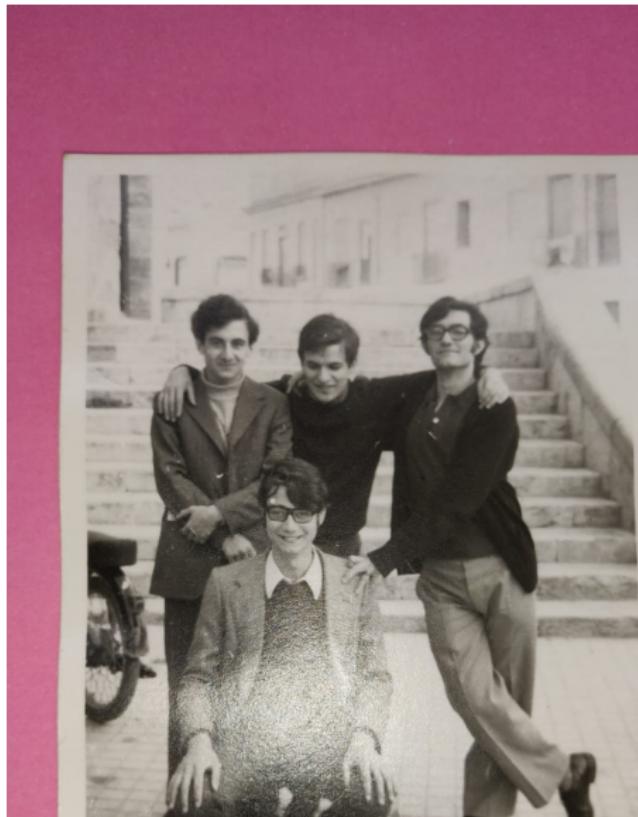
Lemma: (1) $\forall g \in G^\infty(\mathbb{B}) \rightarrow (L^\infty(\mathbb{B}))^*$
 $\|g\|_B \leq \|L^\infty(\mathbb{B})\| \text{ and } L^\infty(\mathbb{B}) \subset \text{dom } g^*$
(2) $\forall g \in L^\infty(\mathbb{T}), L^\infty(\mathbb{B}) \rightarrow \mathbb{C}, L^\infty(\mathbb{B}) \subset \text{dom } g$
 $L^\infty(\mathbb{B}) \text{ and } \|L^\infty(\mathbb{B})\| \leq \|L^\infty(\mathbb{T})\|$
(3) $L: A(\mathbb{D}) \rightarrow \mathbb{C}$ $\cup_{n \in \mathbb{N}} \subset \text{closed ball}$ $L(A(\mathbb{D}))$
 \rightarrow extend to $C(\mathbb{T})$ $\xrightarrow{\text{Hahn-Kakutani}}$ $H\text{-B}$
 $\in \text{dom } L$ $\subset \mathbb{T}$ measure \mathbb{T}
The Riesz theorem \exists no point set $A(\mathbb{D})$



"YOU WANT PROOF? I'LL GIVE YOU PROOF!"















9/6/25, 13:09

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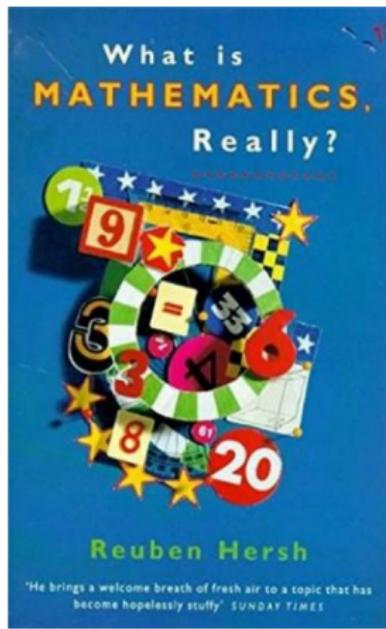






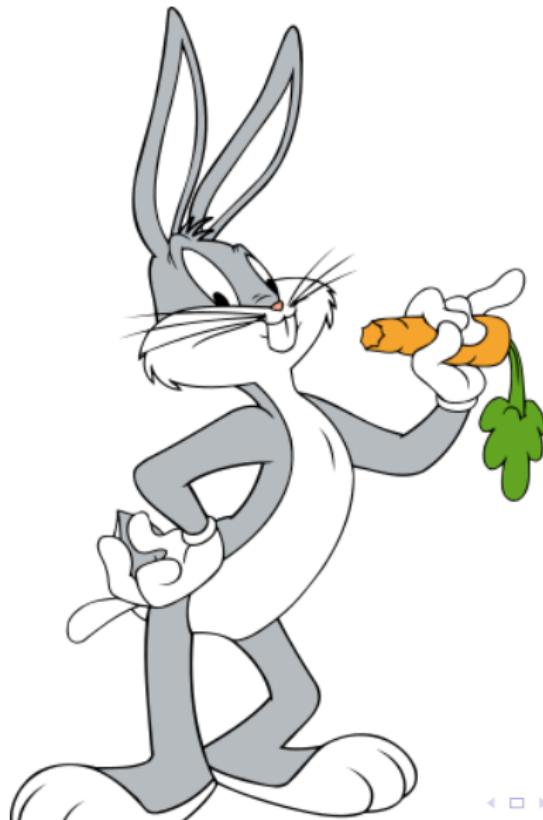














For he's a jolly good fellow, for he's a jolly good fellow
For he's a jolly good fellow, which nobody can deny!