w^*-w-points of continuity of the dual unit ball of a Banach space

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Functional Analysis Meeting on the occasion of the 70th birthday of José Bonet





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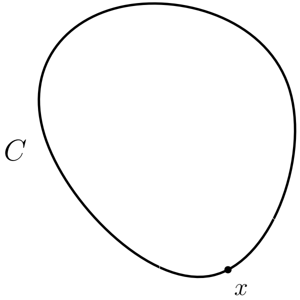
Outline

1. Why points of w^*-w-continuity?
2. Points of w^*-w-continuity and Hahn–Banach extensions
3. Stability
4. M-embedded spaces

Denting points



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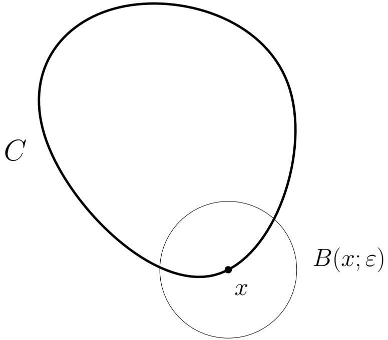

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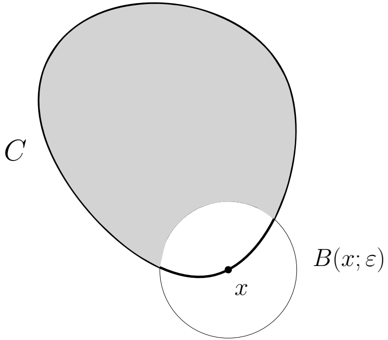

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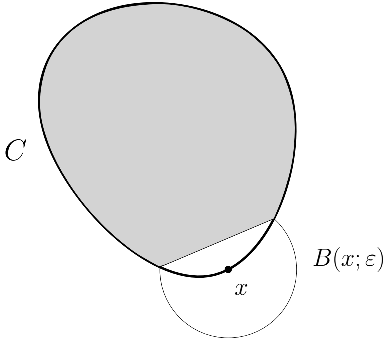

Denting points

Theorem (Lin–Lin–Troyanski'88)

Denting point $\Leftrightarrow$ *extremal* + w-$\|\cdot\|$-*continuity.*

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Denting point $\Leftrightarrow$ *extremal* $+$ w-$\|\cdot\|$-*continuity.*

Theorem (Rieffel'67)

C *convex closed.*
C *RNP* $\Leftrightarrow$ $D = \overline{co}$(*denting points of* D), $\forall D \subset C$ *convex closed.*

Locating the points of w^*-w-continuity

X nonreflexive Banach space

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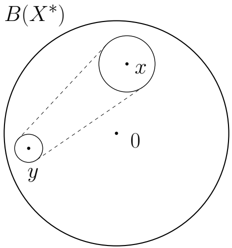

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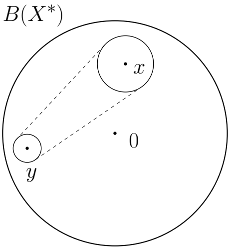

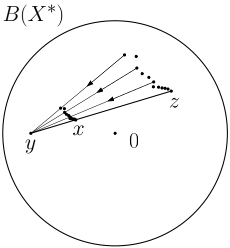

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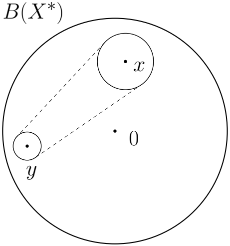

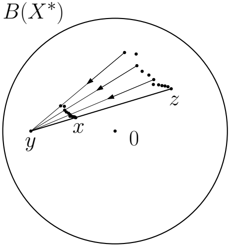

Thus, the points of w^*-w-continuity are on the sphere.

w^*-w-continuity, Hahn–Banach extensions, and renormings

Lemma (G. Godefroy'81)

$x^* \in S(X^*)$. TFAE:

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- *$id : (B(X^*), w^*) \to (B(X^*), w)$ continuous at all $x^* \in S(X^*)$.*

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Exercise: Prove that then X is Asplund

Raja's renorming theorem

Theorem (M. Raja'02)

All $x^* \in S(X^*)$ of w^*-w-continuity.

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Raja's renorming theorem

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All $x^* \in S(X^*)$ of w^*-w-continuity. $\Longrightarrow \exists$ equivalent $|||\cdot|||$ on X^* dual and LUR (and so $|||\cdot|||$ on X is Fréchet, and X is Asplund).

Stability when going up

Theorem

X^{**}/X reflexive. If $x^* \in S(X^*)$ of continuity of $id : (B(X^*), w^*) \to (B(X^*), w)$, then of continuity in all odd higher duals.

Stability when going up

Theorem

X^{**}/X reflexive. If $x^* \in S(X^*)$ of continuity of $id : (B(X^*), w^*) \to (B(X^*), w)$, then of continuity in all odd higher duals. You can add "and x^* extreme point".

No stability when going up

Theorem

X^{**}/X nonreflexive separable. Then $\exists$ renorming Z of X and $x^* \in S(Z^*)$ of continuity for $id : (B(Z^*), w^*) \to (B(Z^*), w)$ and not for $id : (B(Z^{***}), w^*) \to (B(Z^{***}), w)$

No stability when going down

Theorem

$X = L^1[0,1]$. Then id : $(B(X^{(IV)}), w^*) \to (B(X^{IV}), w)$ has an extreme point of continuity, but id : $(B(X^{**}), w^*) \to (B(X^{**}), w)$ is nowhere continuous.

M-embedded spaces

M-embedded space X means M-ideal in X^{**}.

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Example: c_0 in ℓ^∞.

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- *$id : (B(X^{***}), w^*) \to (B(X^{***}), w)$ continuous precisely at $S(X^*)$.*

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Corollary

The set of points of continuity of $id : B(\mathcal{L}(\ell^p)^*, w^*) \to (B(\mathcal{L}(\ell^p)^*, w)$ is precisely $S(\mathcal{K}(\ell^p)^*)$.

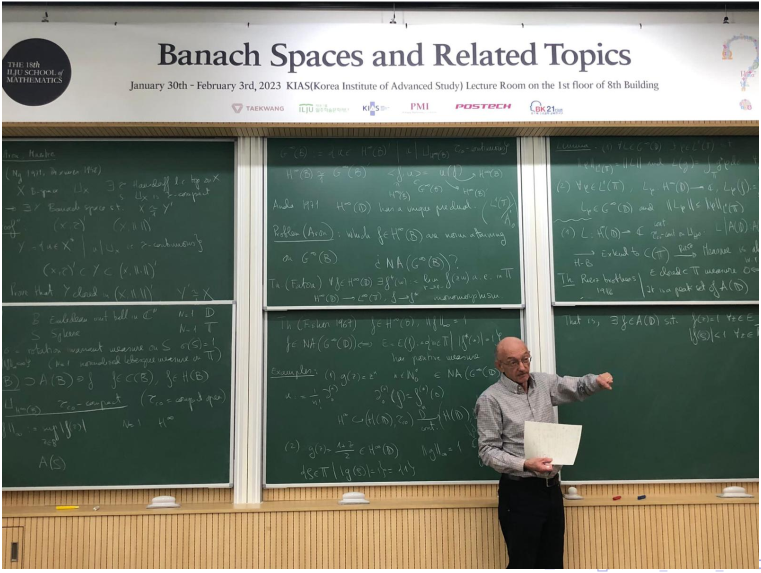

What am I doing?
What the hell is he doing?
what is he doing?

References I

M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler.
Banach Space Theory: the Basis for Linear and Non-Linear Analysis
Springer-Verlag, New York, 2011.

S. Daptari, V. Montesinos, and T.S.S.R.K. Rao
A study of points of w^*-w-continuity in the dual unit ball of a Banach space.
To appear in RACSAM.

"IT'S MOST EXTRAORDINARY ---
AND HE IS ONLY 70."


THE 18th ILJU SCHOOL of MATHEMATICS
Banach Spaces and Related Topics
January 30th - February 3rd, 2023 KIAS(Korea Institute of Advanced Study) Lecture Room on the 1st floor of 8th Building
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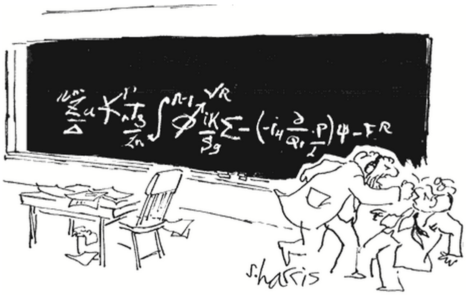
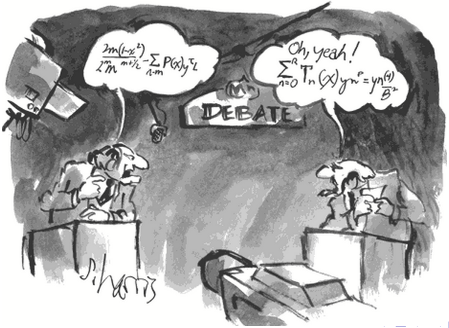
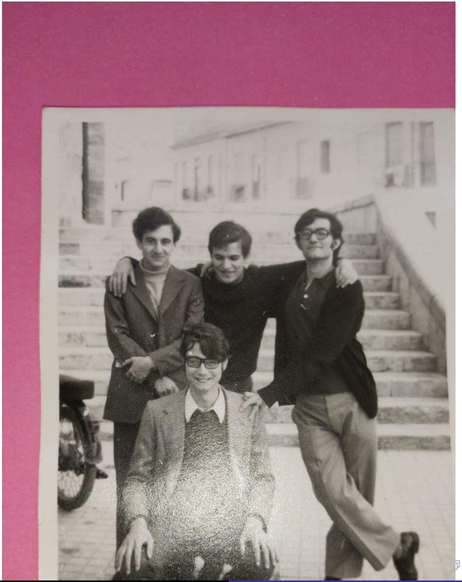
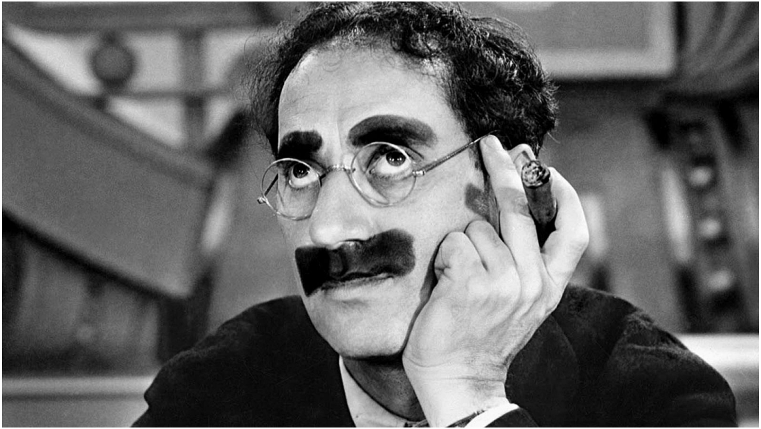


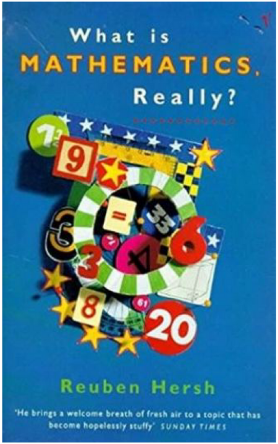
"YOU WANT PROOF? I'LL GIVE YOU PROOF!"


Oh, yeah!
DEBATE




VALENCIA




What is
MATHEMATICS,
Really?
Reuben Hersh
'He brings a welcome breath of fresh air to a topic that has become hopelessly stuffy' SUNDAY TIMES

For he's a jolly good fellow, for he's a jolly good fellow
For he's a jolly good fellow, which nobody can deny!