Compactness of the Weyl operator in $\mathcal{S}_\omega$

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Some previous results

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- Fernández, Galbis (2006) characterized the compactness of the localization operator in L^2 in terms of the symbol
- Boggiatto (2003): if the symbol is in L^∞ and vanishes at infinity the localization operator is compact in $M^{p,q}$
- Bényi, Gröchenig, Heil, Okoudjou (2005): if the simbol is in the Sjöstrand class $M^{\infty,1}$ and can be approximated by functions in the Schwartz class, then the Weyl operator is compact in $M^{p,q}$
- Fernández, Galbis (2007): if the Weyl operator is compact in $M^{p,q}$, then its symbol is in M^∞ and is the limit of a sequence of functions in the Schwartz class (not true for p or q equal 1 or ∞)
- Fernández, Galbis, Primo (2019): extend and improve previous results characterizing the compactness of Fourier integral operators in terms of Gabor matrices
- Boiti, De Martino (2022): sufficient conditions for compactness of localization operators in modulation spaces with exponential weights

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- Obtain consequences to study the compactness of the localization operator in $\mathcal{S}_\omega$.

Notation

- The *Fourier transform* of $f \in L^1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-it\cdot\xi} f(t)dt, \qquad \xi \in \mathbb{R}^d.$$

- The inverse of the Fourier transform is given by

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(\xi)d\xi, \qquad x \in \mathbb{R}^d.$$

- When $f \in L^1(\mathbb{R}^{2d})$, we can define *partial Fourier transforms*. For example, the second partial Fourier transform is

$$\mathcal{F}_2(f)(x,\xi) := \mathcal{F}_{t\mapsto\xi}(f)(x,t) = \int_{\mathbb{R}^d} e^{-it\cdot\xi} f(x,t)dt, \ \ (x,\xi) \in \mathbb{R}^{2d}.$$

- The definitions of Fourier and partial Fourier transform can be extended to more general spaces of distributions.

Weight functions

A non-quasianalytic weight function $\omega : [0, +\infty[\to [0, +\infty[$ is an increasing and continuous function satisfying:

(α) There exists $L \geq 1$ such that $\omega(2t) \leq L\omega(t) + L$, $t \geq 0$;

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{1+t^2} dt < +\infty$;

(γ) $log(t) = O(\omega(t))$ as $t \to \infty$;

(δ) The function $\varphi_\omega(t) = \omega(e^t)$ is convex.

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- Extended to $\mathbb{C}^d$ radially: $\omega(\xi) = \omega(|\xi|)$, being $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{C}^d$ and $|\xi| = \sqrt{|\xi_1|^2 + \cdots + |\xi_d|^2}$.

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- The *Young conjugate* $\varphi^*_\omega : [0, +\infty[\to [0, +\infty[$ *of* φ_ω:

$$\varphi^*_\omega(t) = \sup_{s \geq 0}\{st - \varphi_\omega(s)\}.$$

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Observe that condition (α) is weaker than subadditivity:

$$(\alpha') \qquad \omega(s+t) \leq \omega(s) + \omega(t), \quad s, t \geq 0.$$

Notation II

The space $\mathcal{S}_\omega(\mathbb{R}^d)$: for a weight function ω, it is the space of functions $f \in L^1(\mathbb{R}^d)$ such that $(f, \widehat{f} \in C^\infty(\mathbb{R}^d)$ and) for all $\lambda > 0$ and $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x \in \mathbb{R}^d} |D^\alpha f(x)| e^{\lambda\omega(x)} < +\infty \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^d} |D^\alpha \widehat{f}(\xi)| e^{\lambda\omega(\xi)} < +\infty.$$

We denote by $\mathcal{S}'_\omega(\mathbb{R}^d)$ the strong dual space of $\mathcal{S}_\omega(\mathbb{R}^d)$.

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We denote by $\mathcal{S}'_\omega(\mathbb{R}^d)$ the strong dual space of $\mathcal{S}_\omega(\mathbb{R}^d)$.

- The *translation*, *modulation*, and *phase-shift operators*, for $z = (x, \xi) \in \mathbb{R}^{2d}$, are

$$T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{it\cdot\xi} f(t),$$

$$\Pi(z) f(t) = M_\xi T_x f(t) = e^{it\cdot\xi} f(t - x).$$

- For a window function $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, the STFT of $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$:

$$V_\psi f(z) := \langle f, \Pi(z)\psi \rangle = \int f(y) \overline{\psi(y - x)} e^{-iy\cdot\xi} dy.$$

Gröchenig-Zimmermann (2004)

If $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$, then $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ if and only if

$$\sup_{z \in \mathbb{R}^{2d}} |V_\psi f(z)| e^{\lambda \omega(z)} < +\infty, \qquad \lambda > 0.$$

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- The *cross-Wigner transform* $\mathrm{Wig}(g,f)$, denoting $\mathcal{I}f(t) = f(-t)$, is

$$\mathrm{Wig}(g,f)(z) := 2^d e^{2ix\cdot\xi}\langle g, \Pi(2z)\mathcal{I}f\rangle = \int_{\mathbb{R}^d} g(x+y/2)\overline{f(x-y/2)}e^{-iy\cdot\xi}dy.$$

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- The *Wigner-like transform of* $f \in \mathcal{S}_\omega(\mathbb{R}^{2d})$:

$$\mathrm{Wig}[f](z) := \mathcal{F}_2(\mathcal{T}f)(z) = \int_{\mathbb{R}^d} f(x+y/2, x-y/2)e^{-iy\cdot\xi}dy,$$

where $\mathcal{T}f(x,y) = f(x+y/2, x-y/2)$.

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  where $\mathcal{T}f(x,y) = f(x+y/2, x-y/2)$.
- $\mathrm{Wig}[f]$ is invertible in $\mathcal{S}_\omega(\mathbb{R}^{2d})$ and in $\mathcal{S}'_\omega(\mathbb{R}^{2d})$ and, moreover,

$$\mathrm{Wig}^{-1}[f] = \mathcal{T}^{-1}\mathcal{F}_2^{-1}(f).$$

The Weyl operator

For $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$, the *Weyl operator* $a^w(x, D)$ applied to $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ is the distribution defined by

$$\langle a^w(x, D)f, g\rangle = \frac{1}{(2\pi)^d}\langle a, \mathrm{Wig}(g, f)\rangle, \qquad g \in \mathcal{S}_\omega(\mathbb{R}^d).$$

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- The Weyl operator can be realized as

$$a^w(x,D)f = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a\Big(\frac{x+y}{2},\xi\Big) f(y)\,dy\,d\xi,$$

for instance if $f \in \mathcal{S}_\omega(\mathbb{R}^d)$, $a \in \mathcal{S}_\omega(\mathbb{R}^{2d})$.

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  for instance if $f \in \mathcal{S}_\omega(\mathbb{R}^d)$, $a \in \mathcal{S}_\omega(\mathbb{R}^{2d})$.
- Sufficient conditions on a such that

$$a^w(x,D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$$

  continuously are given by Prangoski and Asensio-Jornet.

Compact and bounded operators

A linear operator $T : E \to F$ between two locally convex spaces E and F is said to be *compact* (resp. *bounded*) if there exists an (open) 0-neighbourhood U in E such that $T(U)$ is relatively compact (resp. bounded) in F.

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Since $\mathcal{S}_\omega$ is a nuclear Fréchet space, it is Montel. Hence, every bounded set is rel. compact. So, $T : \mathcal{S}_\omega \to \mathcal{S}_\omega$ is bounded if and only if it is compact.

Lemma (Albanese, Bonet, Ricker (2019))

Let $X := \mathrm{proj}_m X_m$ and $Y := \mathrm{proj}_m Y_m$ be Fréchet spaces such that $X = \cap_{m\in\mathbb{N}} X_m$ with each $(X_m, \|\cdot\|_m)$ a Banach space and $Y = \cap_{m\in\mathbb{N}} Y_m$ with each $(Y_m, \|\cdot\|_m)$ a Banach space. Moreover, it is assumed that X is dense in X_m and $X_{m+1} \subseteq X_m$ with a continuous inclusion for each $m \in \mathbb{N}$, and $Y_{m+1} \subseteq Y_m$ with a continuous inclusion for each $m \in \mathbb{N}$. Let $T : X \to Y$ be a linear operator.

(i) *T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that T has a unique continuous linear extension $T_{m,n} : X_m \to Y_n$.*

(ii) *Assume that T is continuous. Then T is a bounded operator if and only if there exists $k_0 \in \mathbb{N}$ such that for every $M \in \mathbb{N}$ the operator T has a unique linear and continuous extension $T_{k_0,M} : X_{k_0} \to Y_M$.*

Modulation spaces with exponential weights

- ω subadditive $\quad m_\lambda(x,\xi) = e^{\lambda\omega(x,\xi)}, \quad \lambda \in \mathbb{R}, \quad (x,\xi) \in \mathbb{R}^{2d}$.

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- $L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})$ $(1 \leq p, q < +\infty)$ space of measurable functions $F : \mathbb{R}^{2d} \to \mathbb{C}$ such that

$$\left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,\xi)|^p e^{\lambda p \omega(x,\xi)} dx \right)^{q/p} d\xi \right)^{1/q} < +\infty,$$

with standard extensions when $p = +\infty$ or $q = +\infty$.

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  with standard extensions when $p = +\infty$ or $q = +\infty$.
- The modulation space is defined as

$$M^{p,q}_\lambda(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'_\omega(\mathbb{R}^d) \ : \ V_\psi f \in L^{p,q}_{m_\lambda}(\mathbb{R}^{2d}) \right\},$$

  endowed with the norm $\|f\|_{M^{p,q}_\lambda(\mathbb{R}^d)} = \|V_\psi f\|_{L^{p,q}_{m_\lambda}(\mathbb{R}^{2d})}$.

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Theorem

The space $M^{p,q}_\lambda(\mathbb{R}^d)$ is Banach and, we have

$$\mathcal{S}_\omega(\mathbb{R}^d) = \bigcap_{\lambda > 0} M^{p,q}_\lambda(\mathbb{R}^d).$$

Moreover, $\mathcal{S}_\omega(\mathbb{R}^d)$ is a dense subspace of $M^{p,q}_\lambda(\mathbb{R}^d)$ for $1 \le p, q < +\infty$ and the inclusion $M^{p,q}_\lambda(\mathbb{R}^d) \hookrightarrow M^{p,q}_\mu(\mathbb{R}^d)$ is continuous for all $\lambda > \mu$.

Continuity and compactness of the Weyl operator

Proposition

Let $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$. *The Weyl operator satisfies* $a^w(x,D)\left(\mathcal{S}_\omega(\mathbb{R}^d)\right) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$ *if and only if* $a^w(x,D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ *is continuous.*

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Proposition

Assume that T is a linear and continuous map from $\mathcal{S}_\omega(\mathbb{R}^{d_1})$ into $\mathcal{S}'_\omega(\mathbb{R}^{d_2})$ and fix $\lambda, \mu > 0$. The operator T extends (uniquely) to a continuous mapping from $M^1_\mu(\mathbb{R}^{d_1})$ into $M^\infty_\lambda(\mathbb{R}^{d_2})$ if and only if the kernel $K \in \mathcal{S}'_\omega(\mathbb{R}^{d_2+d_1})$ of T satisfies

$$\sup_{(x,y,\xi,\eta)\in\mathbb{R}^{2(d_2+d_1)}} |V_\psi K(x,y,\xi,\eta)| e^{\lambda\omega(x,\xi)} e^{-\mu\omega(y,\eta)} < +\infty,$$

for any $\psi \in \mathcal{S}_\omega(\mathbb{R}^{d_2+d_1}) \setminus \{0\}$.

Continuity and compactness of the Weyl operator II

When $T = a^w(x, D)$, the Weyl operator, its kernel K is given in terms of the symbol a by

$$K(x,y) = \mathcal{F}^{-1}_{\xi \mapsto x-y}(a)\Big(\frac{x+y}{2}, \xi\Big).$$

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But, as is easy to see $K = \mathrm{Wig}^{-1}[a]$. Hence, we have

Theorem

Let ω be subadditive and $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$.

(i) The Weyl operator satisfies $a^w(x, D)\left(\mathcal{S}_\omega(\mathbb{R}^d)\right) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$ (and, hence, $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is continuous) if and only if for every $\lambda > 0$ there exists $\mu > 0$: $\mathrm{Wig}^{-1}[a] \in M^\infty_{\lambda \otimes (-\mu)}(\mathbb{R}^{2d})$.

(ii) The Weyl operator $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is compact if and only if there exists $\mu > 0$ such that $\mathrm{Wig}^{-1}[a] \in \bigcap_{\lambda > 0} M^\infty_{\lambda \otimes (-\mu)}(\mathbb{R}^{2d})$.

Continuity and compactness of the Weyl operator III

Theorem

Let ω be a subadditive weight function, $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^{2d})$, and $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$.

(i) The Weyl operator satisfies $a^w(x, D)\left(\mathcal{S}_\omega(\mathbb{R}^d)\right) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$ (and, hence, $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is continuous) if and only if for every $\lambda > 0$, there exists $\mu > 0$ such that

$$\sup_{x,\xi,\eta,y\in\mathbb{R}^d} |V_\psi a(x, \xi, \eta, y)| e^{\lambda\omega(x-y/2,\xi+\eta/2)} e^{-\mu\omega(x+y/2,-\xi+\eta/2)} < +\infty.$$

(ii) The Weyl operator $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is compact if and only if there exists $\mu > 0$ such that for every $\lambda > 0$, the last supremum is finite.

Multipliers and convolutors in $\mathcal{S}_\omega(\mathbb{R}^d)$. Some examples

Studied by Albanse and Mele in 2021:

- Space of ω-*multipliers* in $\mathcal{S}_\omega(\mathbb{R}^d)$, $O_M^\omega(\mathbb{R}^d)$: functions $F \in C^\infty(\mathbb{R}^d)$ s.t. $F \cdot f \in \mathcal{S}_\omega(\mathbb{R}^d)$ for every $f \in \mathcal{S}_\omega(\mathbb{R}^d)$.
- Space of ω-*convolutors in* $\mathcal{S}_\omega(\mathbb{R}^d)$, $(O_C^\omega)'(\mathbb{R}^d)$: those $a \in \mathcal{S}'_\omega(\mathbb{R}^d)$ s.t. $a * f \in \mathcal{S}_\omega(\mathbb{R}^d)$ for every $f \in \mathcal{S}_\omega(\mathbb{R}^d)$.

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Generalizing Bargetz-Ortner, 2014:

Theorem

Given $0 \neq \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$, we have:

- $F \in C^\infty(\mathbb{R}^d)$, $F \in O_M^\omega(\mathbb{R}^d)$ if and only if for all $\lambda > 0$ there exist $C_\lambda, \mu_\lambda > 0$ such that

$$|V_\psi F(x,\xi)| \leq C_\lambda e^{-\lambda\omega(\xi)} e^{\mu_\lambda \omega(x)}, \qquad (x,\xi) \in \mathbb{R}^{2d}.$$

- $a \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $a \in (O_C^\omega)'(\mathbb{R}^d)$ if and only if for every $\mu > 0$ there exist $C_\mu, \lambda_\mu > 0$ such that

$$|V_\psi a(x,\xi)| \leq C_\mu e^{-\mu\omega(x)} e^{\lambda_\mu \omega(\xi)}, \qquad (x,\xi) \in \mathbb{R}^{2d}.$$

Examples

Multipliers

- We consider $a = a(x)$ with $a \in \mathcal{S}'_\omega(\mathbb{R}^d)$, we have $a^w(x) = M_a$.
- $a^w(x) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ continuous $\Leftrightarrow$ $a \in O_M^\omega(\mathbb{R}^d)$
- $a^w(x) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ compact $\Leftrightarrow$ $a = 0$

Convolutors, for weights satisfying $\log(t) = o(\omega(t))$, $t \to \infty$

- Now, we take $a = a(\xi) \in \mathcal{S}'_\omega(\mathbb{R}^d)$; it is easy to see that

$$a^w(D)f = \mathcal{F}^{-1}(a) * f.$$

- $a^w(D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ continuous $\Leftrightarrow$ $\mathcal{F}^{-1}(a) \in (O_C^\omega)'(\mathbb{R}^d)$ $\Leftrightarrow$ $a \in O_M^\omega(\mathbb{R}^d)$
- $a^w(D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ compact $\Leftrightarrow$ $a = 0$

Examples II. Continuity is not characterized by multipliers

- Consider $a(x,\xi) = \delta_x \otimes a_2(\xi)$ with $a_2 \in (O_C^\omega)'(\mathbb{R}^d)$
- $a^w(x,D)f(t) = 2^d \mathcal{F}^{-1}(a_2)(2t)f(-t)$.
- since $a_2 \in (O_C^\omega)'(\mathbb{R}^d)$, we have $\mathcal{F}^{-1}(a_2) \in O_M^\omega(\mathbb{R}^d)$ and hence $a^w(x,D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ continuously
- However, $a = \delta_x \otimes a_2(\xi) \notin O_M^\omega(\mathbb{R}^{2d})$ for every $0 \neq a_2 \in (O_C^\omega)'(\mathbb{R}^d)$

- $a(x,\xi) = a_1(x) \otimes \delta_\xi$ with $a_1 \in (O_C^\omega)'(\mathbb{R}^d)$
- $a^w(x,D)f = \Lambda_2 a_1 * \mathcal{I}f$, where $\Lambda_2 a_1(x) = a_1(x/2)$ and $\mathcal{I}f(t) = f(-t)$
- So $a^w(x,D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ continuously
- however, $a(x,\xi)$ is not an ω-multiplier for $0 \neq a_1 \in (O_C^\omega)'(\mathbb{R}^d)$

Examples III. Continuity is not characterized by multipliers

- Consider $a(x,\xi) = e^{-2ix\cdot\xi} f(\xi)$ for some $f \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$
- Then $a \in O_M^\omega(\mathbb{R}^{2d})$
- the Weyl operator $a^w(x,D)$ is not well defined in $\mathcal{S}_\omega(\mathbb{R}^d)$

Indeed, for $g \in \mathcal{S}_\omega(\mathbb{R}^d)$,

$$\begin{aligned} a^w(x,D)g &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-s)\cdot\xi} a\Big(\frac{x+s}{2},\xi\Big) g(s)\,ds\,d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-s)\cdot\xi} e^{-2i\frac{x+s}{2}\cdot\xi} f(\xi) g(s)\,ds\,d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{-i(2s)\cdot\xi} f(\xi) g(s)\,ds\,d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \widehat{f}(2s) g(s)\,ds = C_g, \end{aligned}$$

a constant. Since there exist $g \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfying $C_g \neq 0$ (for example, $g(s) = \widehat{f}(2s)$), the operator is not well defined.

Examples IV

Consider that $a = \mathrm{Wig}(g, f)$ for some $f, g \in \mathcal{S}'_\omega(\mathbb{R}^d)$. For every $h, k \in \mathcal{S}_\omega(\mathbb{R}^d)$, we have by Moyal's formula:

$$\langle a^w(x, D)h, k\rangle = (2\pi)^{-d}\langle \mathrm{Wig}(g, f), \mathrm{Wig}(k, h)\rangle = \langle (2\pi)^{-d}\overline{\langle f, h\rangle} g, k\rangle,$$

which means that

$$a^w(x, D)h = (2\pi)^{-d}\overline{\langle f, h\rangle} g.$$

Then, $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is well defined if and only if $g \in \mathcal{S}_\omega(\mathbb{R}^d)$, for every $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$.

Moreover, in this case the operator is also compact from the previous results.

A consequence for localization operators

If $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$ and $\psi, \gamma \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, the localization operator is defined as

$$L^a_{\psi,\gamma} = V^*_\gamma \circ \mathcal{M}_a \circ V_\psi;$$

We have

$$L^a_{\psi,\gamma} = (a * \mathrm{Wig}(\gamma, \psi))^w(x, D).$$

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Theorem

Let ω be subadditive and $a \in \mathcal{S}'_\omega(\mathbb{R}^{2d})$.

(i) The Weyl operator $a^w(x, D) : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is compact.

(ii) $a \in (O^\omega_C)'(\mathbb{R}^{2d})$.

(iii) The localization operator $L^a_{\psi,\gamma} : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is compact for every pair of windows in $\mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$.

We have $(i) \Rightarrow (ii) \Rightarrow (iii)$.

However, the reverse implications are not true.

Thank you for your attention

and...

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Happy birthday Pepe!!!