International Workshop on Functional Analysis

on the Occasion of the 70th Birthday of José Bonet
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Spheres of positive elements as metric invariants

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Musicians (mathematicians) don't retire; they stop when there's no more music in them. – Louis Armstrong.

Problems are often stated in vague terms… because it is quite uncertain what the problems really are. – John von Neumann

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More precisely,

A challenge:

Suppose X and Y are two Banach/normed spaces, such that there is a surjective isometry $\Delta : (\mathcal{S}_X, d_X) \to (\mathcal{S}_Y, d_Y)$. Are X and Y isometrically isomorphic?

[Mazur–Ulam theorem]

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Let X be a real normed space. Any convex body (i.e., a closed convex subset with non-empty interior) is a metric invariant for X. The closed unit ball of X, $\mathcal{B}_X$, is a metric invariant of X.

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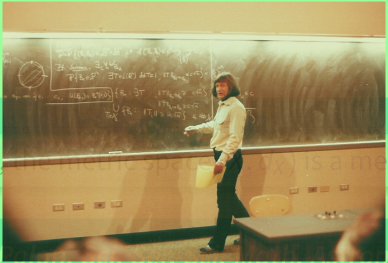
Moreover, if X and Y are normed spaces, and $\Delta : \mathcal{B}_X \to \mathcal{B}_Y$ is a surjective isometry. Then there exists a surjective real linear isometry $T : X \to Y$ extending the original mapping Δ.



Banach Center Photo Archives at KSU

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The non-emptiness of the topological interior of a convex body is crucial in the arguments.

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Caution!!

Tingley's problem remains unsolved even in the simple case of a surjective isometry between the unit spheres of two Banach spaces of dimension ≥ 3.

[T. Banakh, *J. Math. Anal. Appl.*'2021]

The unit sphere of a 2-dimensional Banach space is a metric invariant in the class of 2-dimensional Banach spaces.

Every surjective isometry between the unit spheres of two 2-dimensional Banach spaces extends to a surjective linear isometry between the spaces.

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[Ding, *Science in China*'2002, M.M. Day, *Trans. Amer. Math. Soc.*'1947, Becerra, Cueto, Fernández, Pe., *J. Inst. Math. Jussieu*'2019]

The unit sphere of a Hilbert space is a metric invariant. Moreover, every surjective isometry from the unit sphere of a Hilbert space onto the unit sphere of a Banach space extends to a surjective real linear isometry.

We recall that a C*-algebra is a complex Banach algebra A together with an algebra involution $a \mapsto a^*$ satisfying the Gelfand-Naimark axiom $\|aa^*\| = \|a\|^2$ for all $a \in A$.

A von Neumann algebra is a C*-algebra which is also a dual Banach space. For each complex Hilbert space H, $B(H)$ is a von Neumann algebra.

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[Fernández-Polo, Pe., *J. Math. Anal. Appl.*'2018]

The unit sphere of a von Neumann algebra is a metric invariant in the class of von Neumann algebras.

M and $N \to$ von Neumann algebras, $\Delta : S(M) \to S(N) \to$ surjective isometry. Then Δ extends to a surjective real-linear isometry.

[M. Mori, N. Ozawa, *Studia Math.*'2020]

The unit sphere of a unital C*-algebra or of a real von Neumann algebra is a metric invariant.

$\Delta : S(A) \to S(E) \to$ surjective isometry, $A \to$ unital C*-algebra or a real von Neumann algebra, $E \to$ a real Banach space. Then Δ extends to a surjective real-linear isometry.

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Are there strictly smaller metric invariants?

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Obstacles:

- ✗ The set $\partial_e(\mathcal{B}_X)$ can be empty like in the case of c_0 and $K(H)$ for an infinite dimensional Hilbert space X.
- ✗ We can have $\partial_e(\mathcal{B}_X) = S(X)$, for example, when X is a Hilbert space. This is just a reformulation of Tingley's problem.

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For $X = \mathbb{R} \oplus^{\infty} \mathbb{R}$, we have

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with $d(p_i, p_j) = \|p_i - p_j\| = 2(1 - \delta_{i,j})$, for every $i, j \in \{1, \ldots, 4\}$. The mapping $\Delta : \partial_e(\mathcal{B}_X) \to \partial_e(\mathcal{B}_X)$ defined by $\Delta(p_1) = p_2$, $\Delta(p_2) = p_3$, $\Delta(p_3) = p_4$, and $\Delta(p_4) = p_1$, is a surjective isometry which cannot be extended to a real linear isometry on X.

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[H. Choda, Y. Kijima, and Y. Nakagami, 1969]

A von Neumann algebra M is finite if and only if all the extreme points of its closed unit ball are unitaries (i.e. they satisfy $uu^* = u^*u = 1$), that is, $\partial_e(\mathcal{B}_M) = \mathcal{U}(M) = \{\text{unitaries in } M\}$.

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In general, $\mathcal{U}(M) \subsetneq \partial_e(\mathcal{B}_M)$, even in the case $M = B(H)$ for an infinite dimensional H.

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[O. Hatori, L. Molnar , *J. Math. Anal. Appl.*'2014]

Let W_1 and W_2 be von Neumann algebras. Then every surjective isometry $\Delta : \mathcal{U}(W_1) \to \mathcal{U}(W_2)$ admits a real linear extension to a surjective real linear isometry $T : W_1 \to W_2$.

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Problem:

The conclusion is not completely true for unital C*-algebras essentially because the group of unitaries is not, in general, connected [O. Hatori, *Studia Math.*'2014]

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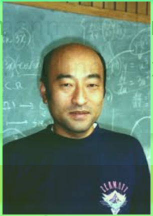

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In words of Alfsen and Shultz *"When a C^*-algebra or a von Neumann algebra is used as an algebraic model of quantum mechanics, then it is only the self-adjoint part of the algebra that represents observables. However, the self-adjoint part of such an algebra is not closed under the given associative product, but only under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Therefore it has been proposed to model quantum mechanics on Jordan algebras rather than associative algebras."*

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During the decade of the thirties in the XXth century, P. Jordan, J. von Neumann, E. Wigner and some other authors introduced the notion of Jordan algebra as a mathematical model for quantum mechanics.

Jordan algebra

A complex *Jordan algebra* M is a (non-necessarily associative) algebra over the complex field whose product (denoted by $\circ$) is abelian and satisfies the so-called *Jordan identity*:

$$(a \circ b) \circ a^2 = a \circ (b \circ a^2), \ \ (a, b \in M).$$

A *Jordan-Banach algebra* is a Jordan algebra M equipped with a complete norm, $\|.\|$, satisfying $\|a \circ b\| \leq \|a\| \, \|b\|$ $(a, b \in M)$.

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[Kaplansky'1976]

A *JB*-algebra* is a complex Jordan-Banach algebra M equipped with an algebra involution "$*$" satisfying an appropriate Gelfand-Naimark axiom: $\|U_a(a^*)\| = \|a\|^3$ for all $a \in M$, where $U_a(b) = 2(a \circ b) \circ b) - b \circ a^2$.

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[Cueto, Pe., *Linear Multilinear Algebra*'2022, Cueto, Enami, Hirota, Miura, Pe., *Linear Algebra Appl.*'2022]

Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be JBW*-algebras. Then every surjective isometry $\Delta : \mathcal{U}(\mathfrak{J}_1) \to \mathcal{U}(\mathfrak{J}_2)$ admits a real linear extension to a surjective real linear isometry $T : \mathfrak{J}_1 \to \mathfrak{J}_2$.

The sphere of positive elements as a metric invariant

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Tingley's problem for positive elements

Suppose X and Y are partially ordered Banach spaces with cones of positive elements denoted by X^+ and Y^+, respectively, having additional "nice-geometric properties". Suppose $\Delta : S(X^+) \to S(Y^+)$ is a surjective isometry. Can we extend Δ to a surjective linear isometry from X onto Y?

This makes sense for many well-known structures, for example every C*-algebra and every JB*-algebra with their cones of positive elements.

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[G. Nagy , *Publ. Math. Debrecen*'2018]

Let H and H' be two finite-dimensional complex Hilbert spaces. Suppose $\Delta : S(B(H)^+) \to S(B(H')^+)$ is a surjective isometry. Then Δ extends to a surjective linear isometry from $B(H)$ onto $B(H')$.

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Let H_1, H_2, H_3 and H_4 be complex Hilbert spaces, where H_3 and H_4 are infinite-dimensional and separable. Then every surjective isometry $\Delta : S(B(H_1)^+) \to S(B(H_2)^+)$ (respectively, $\Delta : S(K(H_3)^+) \to S(K(H_4)^+)$) admits a unique extension to a surjective complex linear isometry from $B(H_1)$ onto $B(H_2)$ (respectively, from $K(H_3)$ onto $K(H_4)$).

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We are not going to enter into the details of the technical arguments, which are somehow complicated. However, there is a tool specially designed to attack this problem which deserves to be commented.

Let E and P be subsets of a Banach space X. We define the *unit sphere around E in P* as the set

$$Sph(E;P) := \{x \in P : \|x - b\| = 1 \text{ for all } b \in E\}.$$

If x is an element in X, we write $Sph(x;P)$ for $Sph(\{x\};P)$. If E is a subset of a C*-algebra A, we shall write $Sph^+(E)$ or $Sph_A^+(E)$ for the set $Sph(E;S(A^+))$. For each element a in A, we shall write $Sph^+(a)$ instead of $Sph^+(\{a\})$.

Let me note that we only need “geometry” and a good knowledge of the sets E and P to “control” the above spheres.

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[Pe., *Adv. Oper. Theory*'2018], [X.Q. Lu, C.K. Ng, *J. Math. Anal. Appl.*'2024]

Let a be a norm-one positive element in a C*-algebra A, and consider the following statements:

(a) a is a projection (i.e., a self-adjoint projection);

(b) $Sph_A^+\left(Sph_A^+(\{a\})\right) = \{a\}$.

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Then $(b) \Rightarrow (a)$. Furthermore $(a) \Leftrightarrow (b)$ when $A = B(H)$ or an atomic von Neumann algebra or $K(H_2)$, where H_2 is an infinite-dimensional and separable complex Hilbert space. Equivalence also holds when A is a type I von Neumann algebra.

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Let M and N be two von Neumann algebras. Then every surjective isometry $\Delta : S(M^+) \to S(N^+)$ extends to a Jordan *-isomorphism from M onto N.

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[Saavedra, Pe. *preprint*'2025]

Let a be a norm-one positive element in a JBW*-algebra $\mathfrak{J}$. Then the following statements are equivalent:

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FunctionalAnalysis 70

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(b) $Sph^+_{\mathfrak{J}}\left(Sph^+_{\mathfrak{J}}(\{a\})\right) = \{a\}$.

[Saavedra, Pe. *preprint*'2025]

Let $\mathfrak{J}$ and $\mathfrak{M}$ be two JBW*-algebras such that the type I_2 part of $\mathfrak{J}$ is atomic. Then every surjective isometry $\Delta : S(\mathfrak{J}^+) \to S(\mathfrak{M}^+)$ extends to a Jordan *-isomorphism from $\mathfrak{J}$ onto $\mathfrak{M}$.

Thanks for spending part of your time listening this talk!!!