Hypercyclic and mixing composition operators on $\mathcal{O}_M(\mathbb{R})$

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Workshop on Functional Analysis on the Occasion of José Bonet's 70th Birthday

Linear dynamics

Linear dynamics

Given a TVS (topological vector space) X and an operator (i.e. a continuous linear map) $T : X \to X$ study the properties of the sequence $(T^n)_{n\in\mathbb{N}}$ of iterates of T, where

$$T^n = \underbrace{T \circ \ldots \circ T}_{n-times}.$$

An operator $T\colon X \to X$ is called hypercyclic if there exists a vector $x \in X$ such that the set (called the orbit of x under T)

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is dense in X.

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Let T be an operator on a separable Fréchet space X. TFAE:

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A Fréchet space is a complete TVS which topology can be generated by a countable family of seminorms.

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- mixing if for any non-empty and open subsets U, V in X there is $N \in \mathbb{N}$ with $T^n(U) \cap V \neq \emptyset$ for every $n \geq N$.

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In general:

$$\text{mixing} \Rightarrow \text{topologically transitive}$$

and

$$\text{hypercyclic} \Rightarrow \text{topologically transitive}$$

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1. When this operator is well-defined?
2. What are the dynamical properties of this operator?

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$$p_n(f) = \max_{x\in[-n,n]} \max_{0\leq i\leq n} \left| f^{(i)}(x) \right|$$

The translation is hypercyclic

Theorem

Let $\psi(x) = x + 1$. The operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is hypercyclic.

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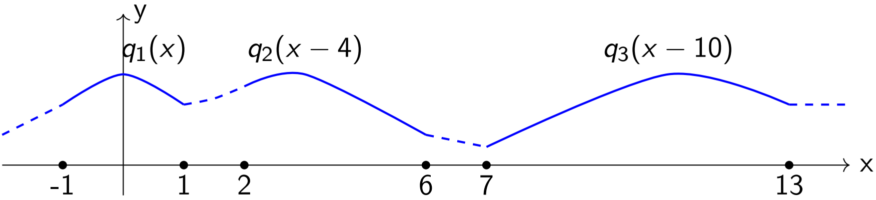

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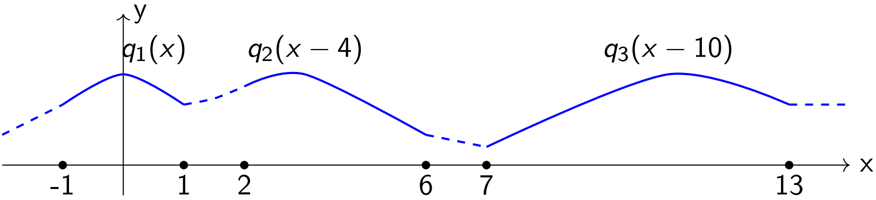

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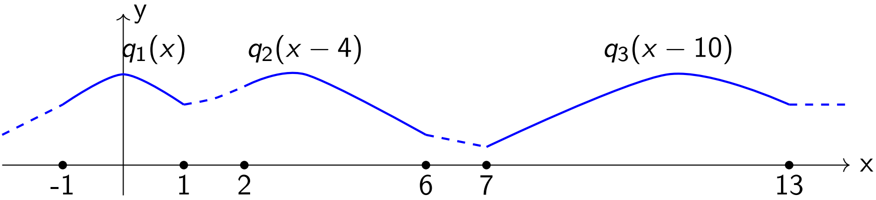

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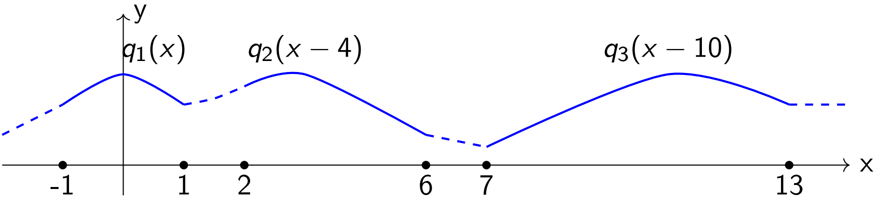

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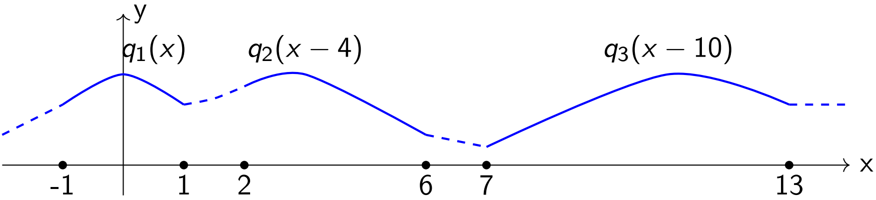

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- $C_\psi^{a_n}(f) = q_n$ on $[-n, n]$

The space of smooth functions

Theorem

For a smooth function $\psi\colon \mathbb{R} \to \mathbb{R}$ TFAE:

1. *For all $x \in \mathbb{R}$ we have that $\psi(x) \neq x$ and $\psi'(x) \neq 0$.*
2. *The operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is hypercyclic.*
3. *The operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is mixing.*

The Schwartz space of rapidly decreasing smooth functions

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This is a Fréchet space, the topology is generated by the sequence of seminorms:

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There are no hypercyclic composition operators acting on $\mathcal{S}(\mathbb{R})$

The space of slowly increasing smooth functions

Joint work with Thomas Kalmes (Chemnitz, Germany)

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A fundamental system of seminorms:

$$p_{m,v}(f) = \sup_{x \in \mathbb{R}} \max_{0 \le j \le m} |v(x) f^{(j)}(x)|, \; f \in \mathcal{O}_M(\mathbb{R}), m \ge 0, v \in \mathcal{S}(\mathbb{R}).$$

Composition operators on $\mathcal{O}_M(\mathbb{R})$

Theorem (Albanese, Jordá, Mele)

TFAE:

1. We have: $\psi \in \mathcal{O}_M(\mathbb{R})$
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Problems

1. Characterize mixing.
2. Does mixing imply hypercyclicity?

First observation

The space $\mathcal{O}_M(\mathbb{R})$ embeds in a continuous and dense way into $C^\infty(\mathbb{R})$, so if C_ψ is hypercyclic (mixing) on $\mathcal{O}_M(\mathbb{R})$, then it is hypercyclic (mixing) on $C^\infty(\mathbb{R})$. In particular:

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- $\psi(x) \neq x$ for $x \in \mathbb{R}$;
- $\psi'(x) \neq 0$ for $x \in \mathbb{R}$.

For $\psi : \mathbb{R} \to \mathbb{R}$:

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- for $n \in \mathbb{N}$: $\psi_n = \underbrace{\psi \circ \ldots \circ \psi}_{n-times}$;
- if ψ is bijective, then for $n \in \mathbb{N}$ the function ψ_{-n} is the inverse of ψ_n.

Hypercyclicity

Theorem

Let $\psi \in \mathcal{O}_M(\mathbb{R})$ be bijective. If

1. $\psi(x) \neq x$ for $x \in \mathbb{R}$,
2. $\psi'(x) \neq 0$ for $x \in \mathbb{R}$,
3. $\{(\psi_n)' : n \in \mathbb{Z}\}$ is bounded in $\mathcal{O}_M(\mathbb{R})$,

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Corollary

For $\psi(x) = x + 1$ the operator $C_\psi : \mathcal{O}_M(\mathbb{R}) \to \mathcal{O}_M(\mathbb{R})$ is hypercyclic.

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2. *There are $a, b \in \mathbb{R}$ such that for every $k \in \mathbb{N}$ and $v \in \mathcal{S}(\mathbb{R})$ we have*

$$\lim_{n\to\infty} \sup_{x\in\psi_{-n}([\min\{a,\psi(a)\},\max\{a,\psi(a)\}])} \left| v(x)(\psi_n)^{(k)}(x) \right| = 0$$

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Can we calculate this?

Examples

Let $\widetilde{\psi} : [0,1] \to \mathbb{R}$ be a smooth function such that: $\widetilde{\psi}(x) = 3x + 1$ for $x \in [0, 1/7]$, $\widetilde{\psi}(x) = 3x - 1$ for $x \in [6/7, 1]$, $\widetilde{\psi}'(x) > 0$ for $x \in [0,1]$.
The function $\psi : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$\psi(x) = \widetilde{\psi}(x - n) + n \quad \text{if} \quad x \in [n, n+1], n \in \mathbb{Z},$$

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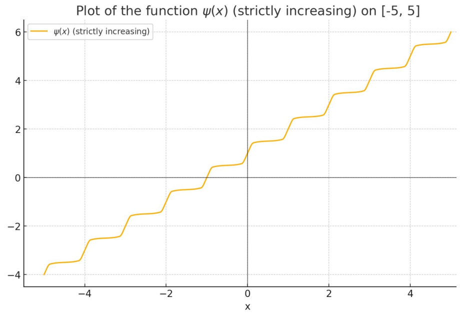

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Hence C_ψ is mixing and hypercyclic.

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Does the mixing property imply the red condition? If yes, then every mixing composition is hypercyclic.

Thanks for your attention!