Smooth extension from closed sets via curve-based criteria

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Der Wissenschaftsfonds.

Frölicher space

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- $f \in \mathcal{F}_X$ if and only if $f \circ c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_X$,
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A map $\varphi : X \to Y$ between Frölicher spaces is called smooth if $\varphi_* \mathcal{C}_X \subseteq \mathcal{C}_Y$. This is equivalent to $\varphi^* \mathcal{F}_Y \subseteq \mathcal{F}_X$ as well as $\mathcal{F}_Y \circ \varphi \circ \mathcal{C}_X \subseteq \mathcal{C}^\infty$.

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Remark

There are other ways of endowing sets X with a synthetic smooth structure, e.g., differential spaces or diffeological spaces.

Theorem

Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ any function. Then:

- f is $\mathcal{C}^\infty$ iff $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. [Boman 1967]

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- f is $\mathcal{C}^{m,\alpha}$ iff $f_*\mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{C}^{m,\alpha}(\mathbb{R}, \mathbb{R})$, where $m \in \mathbb{N}$ and $\alpha \in (0, 1]$. [Boman 1967]

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- f is in the local Zygmund class $\mathcal{Z}^{m,1}$ iff $f_* \mathcal{C}^\infty(\mathbb{R}, U) \subseteq \mathcal{Z}^{m,1}(\mathbb{R}, \mathbb{R})$. [R 2022]

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Remark

It does not work for $\mathcal{C}^m$!

Arc-smooth functions

Let $\emptyset \neq X \subseteq \mathbb{R}^n$ be closed and $\mathcal{C}^\infty(\mathbb{R}, X) := \{c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) : c(\mathbb{R}) \subseteq X\}$.

$$\mathcal{AC}^\infty(X) := \left\{f : X \to \mathbb{R} : f_* \mathcal{C}^\infty(\mathbb{R}, X) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})\right\}$$

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Smoothly extendable functions

$$\mathcal{C}^\infty(X) := \big\{f : X \to \mathbb{R} : \ \exists F \in \mathcal{C}^\infty(\mathbb{R}^n), \ F|_X = f\big\}$$

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Assumption

Let X be closed and fat, i.e., $X = \overline{X^\circ}$.

Theorem [R 2019]

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Hölder sets

Let $\alpha \in (0, 1]$. An α-set is a closed fat subset of $\mathbb{R}^n$ that locally has α-Hölder boundary. A Hölder set is an α-set for some α and a Lipschitz set is a 1-set.

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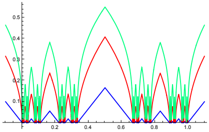
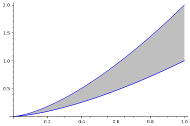
Examples

- Convex closed fat sets are Lipschitz sets.
- The epigraph of $x \mapsto \mathrm{dist}(x, C)^\alpha$, where C is the Cantor set, is an α-set.
- $X = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ x^{3/2} \leq y \leq 2x^{3/2}\}$ is not a Hölder set.





Example

Let $X = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ |y| \le p(x)\}$, where $p : [0,1] \to [0,1]$ is a strictly increasing function satisfying $p(x) \le x^2$ which vanishes to infinite order at 0. The function

$$f : X \to \mathbb{R}, \quad (x,y) \mapsto \sqrt{x^2 + y},$$

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Indeed: Let $x, y : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^\infty$ functions such that $(x(t), y(t)) \in X$ for all $t \in \mathbb{R}$. We show that there is a $\mathcal{C}^\infty$ function $z : \mathbb{R} \to \mathbb{R}$ such that $y = x^2 z$.

Theorem [Joris–Preissmann 1990] If $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are such that $\psi, \varphi\psi \in \mathcal{C}^\infty$ and $|\varphi| \le |\psi|^\alpha$ for some $\alpha > 0$, then $\varphi \in \mathcal{C}^{\lfloor 2\alpha \rfloor}$.

Apply this to $\varphi = y/x^2$ and $\psi = x^2$. Since $|y| \le p(x)$, for each $n \in \mathbb{N}$ there is an interval $[0, \varepsilon_n)$ such that $|\varphi| = |\frac{y}{x^2}| \le |x|^{2n} = |\psi|^n$ if $x \in [0, \varepsilon_n)$.

Example

Let $X = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ |y| \le p(x)\}$ (as on the previous slide) and $Y := \{(x,y) \in \mathbb{R}^2 : x \le 1\} \setminus X^\circ$. Then

$$\mathcal{AC}^\infty(Y) = \mathcal{C}^\infty_{\text{int}}(Y) \neq \mathcal{C}^\infty(Y),$$

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Theorem [R 2022]

Let $X \subseteq \mathbb{R}^n$ be a locally finite union of Hölder sets X_j such that

- if $x \in \partial X$ and $x \in X_i \cap X_j$, then there exists a Hölder set Z such that $x \in Z \subset X_i \cap X_j$.

Then $\mathcal{AC}^\infty(X) = \mathcal{C}^\infty_{\text{int}}(X)$.

Theorem [R 2022]

Let $\alpha, \beta \in (0,1]$ and $m \in \mathbb{N}$. For any α-set $X \subseteq \mathbb{R}^n$,

$$\mathcal{AC}^{mp(\alpha),\beta}(X) \subseteq \mathcal{C}_{\text{int}}^{m,\frac{\alpha\beta}{2q(\alpha)}}(X)$$

where $p(\alpha) := \lceil \frac{2}{\alpha} \rceil$, $q(\alpha) := \lceil \frac{1}{\alpha} \rceil$, and

$$\mathcal{AC}^{m,\beta}(X) := \{f : X \to \mathbb{R} : f_*\mathcal{C}^\infty(\mathbb{R}, X) \subseteq \mathcal{C}^{m,\beta}(\mathbb{R},\mathbb{R})\},$$

$$\mathcal{C}_{\text{int}}^{m,\beta}(X) := \left\{ f : X \to \mathbb{R} : \quad \begin{array}{l} f|_{X^\circ} \in \mathcal{C}^m(X^\circ), \text{ all } f^{(\alpha)}, |\alpha| \le m, \\ \text{extend continuously to } \partial X, \text{ and} \\ \text{all } f^{(\alpha)}, |\alpha| = m, \text{are locally } \beta\text{-Hölder} \end{array} \right\}.$$

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Optimality

- Loss of derivatives: $p(\alpha)$ is optimal.
- Degradation of the Hölder index: optimal for 1-sets. Often the factor $\frac{\alpha}{2q(\alpha)}$ can be replaced by $\frac{1}{2q(\alpha)}$.

Subanalytic sets **[Łojasiewicz 1964]**, **[Hironaka 1973]**

Let M be a real analytic manifold. A subset $X \subseteq M$ is called semianalytic if each $x \in M$ has an open neighborhood U in M such that

$$X \cap U = \bigcup_i \bigcap_j \{f_{ij} = 0,\, g_{ij} > 0\},$$

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A subset $X \subseteq M$ is subanalytic if each $x \in M$ has an open neighborhood U in M such that $X \cap U$ is the projection of a relatively compact semianalytic subset of $M \times N$, where N is another real analytic manifold.
E.g. $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\, x^{3/2} \leq y \leq 2x^{3/2}\}$.

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Example [Osgood 1920s]

The image under $\mathbb{R}^2 \ni (x, y) \mapsto (x, xy, xye^y) \in \mathbb{R}^3$ of the closed unit ball is subanalytic but not semianalytic.

Theorem **[R 2019]**

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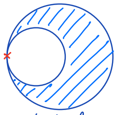
Assumption

We say that X is simple if each $x \in X$ has a basis of neighborhoods U such that $U \cap X^\circ$ is connected.



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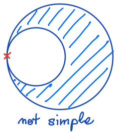

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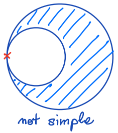

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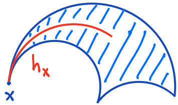
UPC sets

A closed set $X \subseteq \mathbb{R}^n$ is called UPC if there are $M > 0$ and $m, N \in \mathbb{N}_{\geq 1}$ such that for all $x \in X$ there is a polynomial curve $h_x : \mathbb{R} \to \mathbb{R}^n$ of degree $\leq N$ s.t.



- $h_x((0,1]) \subseteq X^\circ$ and $h_x(0) = x$,
- $\mathrm{dist}(h_x(t), \mathbb{R}^n \setminus X) \geq M t^m$ for all $x \in X$ and $t \in [0,1]$.

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Theorem [R 2022]

Let $X \subseteq \mathbb{R}^n$ be a simple compact fat subanalytic set and α a UPC-index of X. Let $m \in \mathbb{N}_{\geq 1}$ and $\beta \in (0,1]$. Then there exists $\ell \in \mathbb{N}_{\geq 1}$ such that

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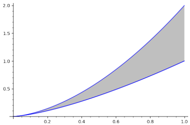
Beyond subanalytic?

$X = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ x^{\sqrt{2}} \leq y \leq x^{\sqrt{2}} + x^2\}$ is not UPC.



Is it true that $\mathcal{AC}^\infty(X) \subseteq \mathcal{C}^\infty(X)$?

Theorem **[Bochnak–Siciak 1970-71]**

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- $f \in \mathcal{C}^\omega(U)$ iff $f \in \mathcal{C}^\infty(U)$ and $f|_\ell$ is real analytic for each affine line ℓ.

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Theorem [Bochnak–Siciak 1970-71]

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Theorem [Bochnak–Kollár-Kucharz 2020]

Let M be a real analytic manifold of dimension $n \geq 3$. Then $f : M \to \mathbb{R}$ is real analytic iff $f|_N$ is real analytic for each real analytic submanifold $N \subseteq M$ homeomorphic to $\mathbb{S}^2$.

Theorem [R 2019], [R 2024]

Let $X \subseteq \mathbb{R}^n$ be a Hölder set or a simple closed fat subanalytic set.

- $f \in \mathcal{C}^\omega(X)$, i.e., f extends to a real analytic function on an open neighborhood of X, iff $f \in \mathcal{C}^\infty(X)$ and $f \circ c \in \mathcal{C}^\omega$ for each germ $c : (\mathbb{R}, 0) \to X$ of a polynomial curve.

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Quantitative versions

The cuspidality of X determines the maximal degree of the polynomial maps needed for testing real analyticity (in first item: $d = 2 \min_{(m,N)} \max\{m, N\}$).

Theorem [Kriegl–Michor–R 2009]

Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Let $\mathcal{E}^{\{*\}}$ be a non-quasianalytic ultradifferentiable class of Roumieu type that is stable under composition. Then $f \in \mathcal{E}^{\{*\}}(U)$ iff $f_* \mathcal{E}^{\{*\}}(\mathbb{R}, U) \subseteq \mathcal{E}^{\{*\}}(\mathbb{R}, \mathbb{R})$.

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Theorem [R 2025]

Let $X \subseteq \mathbb{R}^n$ be a simple closed fat subanalytic set. Let ω be a non-quasianalytic concave weight function satisfying

$$\omega(t^2) = O(\omega(t)) \quad \text{as } t \to \infty,$$

e.g., $\omega_s(t) = ((\log t)_+)^s$ for $s > 1$. Then

$$\begin{aligned} \mathcal{A}\mathcal{E}^{\{\omega\}}(X) &:= \{f : X \to \mathbb{R} : f_* \mathcal{E}^{\{\omega\}}(\mathbb{R}, X) \subseteq \mathcal{E}^{\{\omega\}}(\mathbb{R}, \mathbb{R})\} \\ &= \Big\{ f \in \mathcal{C}^\infty(X) : \forall K \subseteq X \ \exists \rho > 0 \ \sup_{\alpha \in \mathbb{N}^n} \frac{\|\partial^\alpha f\|_K}{e^{-\frac{\varphi^*(\rho|\alpha|)}{\rho}}} < \infty \Big\} \\ &=: \mathcal{E}^{\{\omega\}}(X). \end{aligned}$$

Theorem [R 2025]

- Let $X \subseteq \mathbb{R}^n$ be a simple closed fat subanalytic set. The identities

$$\mathcal{A}\mathcal{C}^\infty(X) = \mathcal{C}^\infty(X), \quad \mathcal{A}\mathcal{C}^\omega(X) = \mathcal{C}^\omega(X), \quad \mathcal{A}\mathcal{E}^{\{\omega\}}(X) = \mathcal{E}^{\{\omega\}}(X),$$

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- They can be lifted to maps with values in a convenient vector space E:

$$\mathcal{A}\mathcal{C}^\infty(X, E) \cong \mathcal{C}^\infty(X, E),$$
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- These spaces are convenient and respective exponential laws hold:

$$\mathcal{AC}^\infty(X_1, \mathcal{AC}^\infty(X_2, E)) \cong \mathcal{AC}^\infty(X_1 \times X_2, E),$$
$$\mathcal{C}^\infty(X_1, \mathcal{C}^\infty(X_2, E)) \cong \mathcal{C}^\infty(X_1 \times X_2, E), \qquad \text{etc.}$$

  where $X_i \subseteq \mathbb{R}^{n_i}$ are simple closed fat subanalytic sets.

On Hölder sets

- Uniform cusp property
- Curve lemmas

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On subanalytic sets

- UPC and the result on Hölder sets
- Łojasiewicz inequality
- L-regular decomposition
- Rectilinearization: If $X \subseteq \mathbb{R}^n$ is compact subanalytic, then there exist finitely many real analytic maps $\varphi_i : \mathbb{R}^n \to \mathbb{R}^n$ such that
  - there are compact $Y_i \subseteq \mathbb{R}^n$, such that $\bigcup_i \varphi_i(Y_i)$ is a neighborhood of X,
  - for each i, $\varphi_i^{-1}(X)$ is a union of quadrants.
- Uniformization: If $X \subseteq \mathbb{R}^n$ is closed subanalytic, then $X = \varphi(N)$ for a proper real analytic map $\varphi : N \to \mathbb{R}^n$ on a real analytic manifold N.

Happy Birthday!