Injectivity and surjectivity of the Stieltjes moment mapping in Gelfand-Shilov classes defined by weight sequences with shifted moments

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Weight sequences

$\mathbb{N}_0 = \{0, 1, 2, ...\}$.

Let $\boldsymbol{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers, with $M_0 = 1$.

$\boldsymbol{M}$ is said to be logarithmically convex or (lc) ((M1) in H. Komatsu (1973)) if

$$M_n^2 \leq M_{n-1}M_{n+1}, \quad n \geq 1.$$

Equivalently, the sequence of quotients of $\boldsymbol{M}$, $\boldsymbol{m} = (m_n := \frac{M_{n+1}}{M_n})_{n\in\mathbb{N}_0}$, is nondecreasing.

We frequently assume that $\boldsymbol{M}$ is (lc) and $\lim_{n\to\infty} m_n = \infty$, and we say $\boldsymbol{M}$ is a weight sequence.

Associated functions

The associated functions with a weight sequence $\boldsymbol{M}$ are defined as

$$\omega_{\boldsymbol{M}}(t) := \sup_{p\in\mathbb{N}_0} \log \frac{t^p}{M_p}, \quad t > 0; \omega_{\boldsymbol{M}}(0) := 0,$$

and

$$h_{\boldsymbol{M}}(t) := \inf_{p\in\mathbb{N}_0} M_p t^p, \quad t > 0;\ h_{\boldsymbol{M}}(0) := 0.$$

They are related by

$$h_{\boldsymbol{M}}(t) = \exp(-\omega_{\boldsymbol{M}}(1/t)), \quad t > 0.$$

Stieltjes moments

Let $\boldsymbol{M}$ be a sequence and $h>0$. $(C_{\boldsymbol{M},h}(0,\infty), s^0_{\boldsymbol{M},h})$ is the Banach space consisting of all $\varphi\in C((0,\infty))$ such that

$$s^0_{\boldsymbol{M},h}(\varphi):=\sup_{p\in\mathbb{N}_0}\sup_{x\in(0,\infty)}\frac{x^p|\varphi(x)|}{h^pM_p}<\infty,\quad \text{i. e. } |\varphi(x)|\le s^0_{\boldsymbol{M},h}(\varphi)h_{\boldsymbol{M}}(h/x).$$

We set $C_{\{\boldsymbol{M}\}}(0,\infty)=\bigcup_{h>0}C_{\boldsymbol{M},h}(0,\infty)$, an (LB) space.
The p-th Stieltjes moment, $p\in\mathbb{N}_0$, of $\varphi\in C_{\{\boldsymbol{M}\}}(0,\infty)$ is defined as

$$\mu_p(\varphi):=\int_0^\infty x^p\varphi(x)\mathrm{d}x.$$

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$$\begin{aligned}|\mu_p(\varphi)|&\le\int_0^{hm_p}x^p|\varphi(x)|\mathrm{d}x+\int_{hm_p}^\infty\frac{x^{p+2}|\varphi(x)|}{x^2}\mathrm{d}x\\&\le s^0_{\boldsymbol{M},h}(\varphi)\left(hm_ph^pM_p+h^{p+2}M_{p+2}\frac{1}{hm_p}\right)\\&=s^0_{\boldsymbol{M},h}(\varphi)h^{p+1}\left(M_{p+1}+\frac{M_{p+2}M_p}{M_{p+1}}\right).\end{aligned}$$

The moment mapping under (dc)

$\boldsymbol{M}$ satisfies (dc) or is derivation closed if for some $C_0 > 0$ and $H > 1$,

$$M_{p+1} \leq C_0 H^p M_p, \quad p \in \mathbb{N}_0.$$

Then

$$\begin{aligned}|\mu_p(\varphi)| &\leq s^0_{\boldsymbol{M},h}(\varphi) h^{p+1} \left(M_{p+1} + \frac{M_{p+2} M_p}{M_{p+1}} \right) \\ &\leq C_0 s^0_{\boldsymbol{M},h}(\varphi) h^{p+1} (H^p M_p + H^{p+1} M_p),\end{aligned}$$

and $(\mu_p(\varphi))_{p \in \mathbb{N}_0}$ belongs to

$$\Lambda_{\{\boldsymbol{M}\}} = \{(c_p)_{p\in\mathbb{N}_0} : \sup_{p\in\mathbb{N}_0} \frac{|c_p|}{h^p M_p} < \infty \text{ for some } h > 0\}.$$

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A standard Stieltjes moment problem in this context consists in the study of the surjectivity and injectivity of the Stieltjes moment mapping $\mathcal{M}$, sending φ to $(\mu_p(\varphi))_{p\in\mathbb{N}_0}$, when defined on $C_{\{\boldsymbol{M}\}}(0,\infty)$ or its subspaces, the Gelfand-Shilov classes, and with target space $\Lambda_{\{\boldsymbol{M}\}}$.

Gelfand-Shilov classes, I

Let $\boldsymbol{M}$ and $\boldsymbol{A}$ be sequences of positive real numbers. For $h > 0$, $(\mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\mathbb{R}), s^{\boldsymbol{A},h}_{\boldsymbol{M},h})$ is the Banach space of all $\varphi \in C^{\infty}(\mathbb{R})$ such that

$$s^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\varphi) := \sup_{p,q\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{|x^p \varphi^{(q)}(x)|}{h^{p+q} M_p A_q} < \infty.$$

We set $\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(\mathbb{R}) = \bigcup_{h>0} \mathcal{S}^{\boldsymbol{A},h}_{\boldsymbol{M},h}(\mathbb{R})$, which is an (LB) space.

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For $h>0$, $\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R})$ consists of all $\varphi\in C^\infty(\mathbb{R})$ such that, for all $q\in\mathbb{N}_0$,

$$s^q_{\boldsymbol{M},h}(\varphi):=\sup_{p\in\mathbb{N}_0}\sup_{x\in\mathbb{R}}\frac{|x^p\varphi^{(q)}(x)|}{h^pM_p}<\infty.$$

$(\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R}),(s^q_{\boldsymbol{M},h})_{q\in\mathbb{N}_0})$ is a Fréchet space.
$\mathcal{S}_{\{\boldsymbol{M}\}}(\mathbb{R})=\bigcup_{h>0}\mathcal{S}_{\boldsymbol{M},h}(\mathbb{R})$ is endowed with its natural (LF) space structure.

Gelfand-Shilov classes, II

We also define

$$\mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(0,\infty) := \{\varphi \in \mathcal{S}^{\{\boldsymbol{A}\}}_{\{\boldsymbol{M}\}}(\mathbb{R}) \mid \operatorname{supp}\varphi \subseteq [0,\infty)\}$$

and

$$\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty) := \{\varphi \in \mathcal{S}_{\{\boldsymbol{M}\}}(\mathbb{R}) \mid \operatorname{supp}\varphi \subseteq [0,\infty)\},$$

with their topologies inherited from their ambient spaces.

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with their topologies inherited from their ambient spaces.

If $\boldsymbol{A}$ satisfies (lc), then $\mathcal{S}_{\{\boldsymbol{M}\}}^{\{\boldsymbol{A}\}}(0,\infty)$ is non-trivial (i. e., it contains non identically zero functions) if and only if $\sum_{p=0}^{\infty} 1/a_p < \infty$, as follows from the Denjoy-Carleman theorem.

Results under (dc) or stronger conditions

A. L. Durán, R. Estrada,, Proc. Amer. Math. Soc. 120 (1994), 529–534.
They prove surjectivity in the Schwartz space, combining the Fourier transform with Borel-Ritt-like theorems from asymptotic analysis.

S.-Y. Chung, D. Kim, Y. Yeom, Fract. Calc. Appl. Anal. 2, 5 (1999), 623–629.
Surjectivity in $\mathcal{S}_{\{(p!^{\alpha})_p\}}(0,\infty)$ (the Gevrey case) whenever $\alpha > 2$.

A. Lastra, J. S., Studia Math. 192 (2009), 111–128.
Surjectivity and (local) right inverses in $\mathcal{S}_{\{(p!M_p)_p\}}(0,\infty)$ for strongly regular sequences $\boldsymbol{M}$ (with moderate growth, stronger than (dc)).

A. Debrouwere, J. Jiménez-Garrido, J. S., RACSAM (2019) 113:3341–3358.
Surjectivity in $\mathcal{S}_{\{(p!M_p)_p\}}(0,\infty)$ and $\mathcal{S}^{\{(p!A_p)_p\}}_{\{(p!M_p)_p\}}(0,\infty)$, moderate growth is frequently weakened into (dc), and injectivity is characterized.

A. Debrouwere, Studia Math. 254 (2020), 295–323.
Complete characterization of the surjectivity and the existence of global right inverses for the moment mapping in Gelfand-Shilov spaces of both Roumieu and Beurling type under (dc), which is essential for his approach.

What can be said without (dc)?

For $\varphi \in C_{\boldsymbol{M},h}(0,\infty)$,

$$|\varphi(x)| \leq s^0_{\boldsymbol{M},h}(\varphi) h_{\boldsymbol{M}}(h/x) = s^0_{\boldsymbol{M},h}(\varphi)\frac{h^{p+1}M_{p+1}}{x^{p+1}}, \quad x \in (hm_p, hm_{p+1});$$

we further split the interval and have

$$\begin{aligned}|\mu_p(\varphi)| &\leq \left(\int_0^{hm_p} + \int_{hm_p}^{hm_{p+1}} + \int_{hm_{p+1}}^{\infty}\right) x^p|\varphi(x)|\mathrm{d}x \\ &\leq s^0_{\boldsymbol{M},h}(\varphi)\Big(hm_p h^p M_p + h^{p+1}M_{p+1}\log\Big(\frac{m_{p+1}}{m_p}\Big) + h^{p+2}M_{p+2}\frac{1}{hm_{p+1}}\Big) \\ &= s^0_{\boldsymbol{M},h}(\varphi)h^{p+1}\Big(M_{p+1} + M_{p+1}\log\Big(\frac{m_{p+1}}{m_p}\Big) + M_{p+1}\Big).\end{aligned}$$

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DEF.: $\boldsymbol{M}$ has shifted moments ((sm) for short) if for some $C_0 > 0$ and $H > 1$,

$$\log(m_{p+1}/m_p) \leq C_0 H^p, \quad p \in \mathbb{N}_0.$$

Then,

$$|\mu_p(\varphi)| \leq s^0_{\boldsymbol{M},h}(\varphi) h^{p+1} \left(2 + C_0 H^p\right) M_{p+1}.$$

A new Stieltjes moment problem

If $\boldsymbol{M}$ satisfies (sm) and we put $\boldsymbol{M}_{+1} := (M_{p+1})_p$, the moment mapping $\mathcal{M} : C_{\{\boldsymbol{M}\}}(0,\infty) \to \Lambda_{\{\boldsymbol{M}_{+1}\}}$ is well-defined and continuous, and a new moment problem can be considered.

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Remarks:

$(\mathrm{dc}) \Leftrightarrow \Lambda_{\{\boldsymbol{M}\}} = \Lambda_{\{\boldsymbol{M}_{+1}\}} \Leftrightarrow \exists C,H>0\colon \forall p,\ m_p \le CH^p$;

$(\mathrm{sm}) \Leftrightarrow \exists C,H>0\colon \forall p,\ \log(m_p) \le CH^p$.

So, (dc) implies (sm), which is strictly weaker.
Moreover, under minimal hypotheses, these conditions are not only sufficient but also necessary for the corresponding moment problems to be well-posed.

Thilliez's index

V. Thilliez (2003) introduces a growth index $\gamma(\boldsymbol{M})$. Now we know:

$$\gamma(\boldsymbol{M}) = \sup\{\gamma > 0 : (m_p/(p+1)^{\gamma})_{p\in\mathbb{N}_0} \text{ is almost increasing}\}$$
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Moreover, $\gamma(\boldsymbol{M}) > 0$ if and only if there exists $B > 0$ such that

$$\sum_{k\geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B\frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0. \qquad [\widehat{\boldsymbol{M}} := (n!M_n)_{n\in\mathbb{N}_0} \text{ has } (M3)]$$

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Examples: $\gamma((\prod_{k=0}^{n}\log^{\beta}(e+k))_{n\in\mathbb{N}_0}) = 0, \quad \beta > 0$

$\gamma\big((n!^{\alpha}\prod_{m=0}^{n}\log^{\beta}(e+m)\big)_{n\in\mathbb{N}_0}) = \alpha, \quad \alpha > 0,\ \beta \in \mathbb{R}$ ($\beta = 0$, Gevrey)

$\gamma((q^{n^{\sigma}})_{n\in\mathbb{N}_0}) = \infty, \quad q > 1,\ \sigma > 1$ ($\sigma = 2$, $q-$Gevrey)

$\gamma((n^{\tau n^{\sigma}})_{n\in\mathbb{N}_0}) = \infty, \quad \tau > 0,\ \sigma > 1$ (Pilipović-Teofanov-Tomić).

Optimal $\boldsymbol{M}$-flat functions

J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl, Results Math. (2023) 78:98.

Proposition

Let $\boldsymbol{M}$ be a weight sequence with $\gamma(\boldsymbol{M}) > 0$. Then, for any $0 < \gamma < \gamma(\boldsymbol{M})$ there exists an optimal $\{\boldsymbol{M}\}-$flat function G in $S_\gamma = \{z \in \mathcal{R} \colon |\arg(z)| < \pi\gamma/2\}$, i.e., $G \in \mathcal{O}(S_\gamma)$ and

(i) $\exists K_3, K_4 > 0 : \quad |G(z)| \le K_3 h_{\boldsymbol{M}}(K_4|z|)$ *for all* $z \in S$, *($\{\boldsymbol{M}\}-$flatness)*

(ii) $\exists K_1, K_2 > 0 : \quad K_1 h_{\boldsymbol{M}}(K_2 x) \le G(x)$ *for all* $x > 0$.

Target space for the Stieltjes moment mapping

If G is an optimal $\{\boldsymbol{M}\}-$flat function in S_γ, we define a kernel function

$$e(z) := G\left(\frac{1}{z}\right), \quad z \in S_\gamma.$$

We have $K_1 h_{\boldsymbol{M}}\left(\frac{K_2}{x}\right) \le e(x) \le K_3 h_{\boldsymbol{M}}\left(\frac{K_4}{x}\right) \implies e \in C_{\{\boldsymbol{M}\}}(0,\infty)$.

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Proposition (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

Suppose $\boldsymbol{M}$ is a weight sequence with $\gamma(\boldsymbol{M}) > 0$. Then,

- *$\boldsymbol{M}$ satisfies (dc) if and only if $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \subset \Lambda_{\{\boldsymbol{M}\}}$ if and only if for every $\boldsymbol{M}-$flat function G one has $(\mu_p(e))_p$ is equivalent to $\boldsymbol{M}$.*
- *$\boldsymbol{M}$ satisfies (sm) if and only if $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \subset \Lambda_{\{\boldsymbol{M}_{+1}\}}$ if and only if for every $\boldsymbol{M}-$flat function G one has $(\mu_p(e))_p$ is equivalent to $\boldsymbol{M}_{+1}$.*

The Fourier transform on Gelfand-Shilov classes

The Fourier transform is considered in the form

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x) e^{ix\xi} \mathrm{d}x, \qquad \varphi \in L^1(\mathbb{R}).$$

Proposition (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

Let $\boldsymbol{M}$ be a weight sequence satisfying (sm), and $\boldsymbol{A}$ be either an almost increasing sequence, or a sequence such that $\liminf_{p\to\infty} A_p^{1/p} > 0$ and $\widehat{\boldsymbol{A}}$ satisfies (lc). Then

(i) $\mathcal{F}\colon \mathcal{S}_{\{\boldsymbol{M}\}}^{\{\widehat{\boldsymbol{A}}\}}(\mathbb{R}) \to \mathcal{S}_{\{\widehat{\boldsymbol{A}}\}}^{\{\boldsymbol{M}_{+1}\}}(\mathbb{R})$ *is continuous, and the same holds at the level of Banach spaces with a uniform scaling of the value h.*

(ii) *The previous statement is valid also for $\mathcal{F}^{-1}$.*

Ultraholomorphic classes in a half-plane

We write $\mathbb{H}$ for the open upper half-plane of $\mathbb{C}$. For $h>0$,

$$\mathcal{A}_{\boldsymbol{M},h}(\mathbb{H}) := \{f \in \mathcal{O}(\mathbb{H}) \colon \|f\|_{\boldsymbol{M},h} := \sup_{p\in\mathbb{N}_0} \sup_{z\in\mathbb{H}} \frac{|f^{(p)}(z)|}{h^p M_p} < \infty\}.$$

$(\mathcal{A}_{\boldsymbol{M},h}(\mathbb{H}), \|\cdot\|_{\boldsymbol{M},h})$ is a Banach space, and $\mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H}) := \bigcup_{h>0} \mathcal{A}_{\boldsymbol{M},h}(\mathbb{H})$ an (LB) space.

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The asymptotic (Peano-)Borel mapping

$$\mathcal{B} : \mathcal{A}_{\{\boldsymbol{M}\}}(\mathbb{H}) \to \Lambda_{\{\boldsymbol{M}\}}, \quad f \mapsto (f^{(p)}(0))_p,$$

is linear and continuous.

Auxiliary function and result

We say $\boldsymbol{A}$ satisfies the condition (nq) if $\sum_{p=0}^{\infty} \frac{1}{(p+1)a_p} < \infty$.

Lemma (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

Let $\boldsymbol{M}$ be a sequence satisfying (lc) *and* $\liminf_{p\to\infty}(M_p/p!)^{1/p} > 0$, *and let* $\boldsymbol{A}$ *satisfy* (nq), *and such that* $\widehat{\boldsymbol{A}}$ *is a weight sequence.*
Then, there is $G_0 \in \mathcal{O}(\{\mathrm{Im}(z) > -1\})$ *(A. Debrouwere, J. Jiménez-Garrido, J. S. (2019)), which does not vanish at any point, and such that the map*

$$f \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H}) \to (fG_0)|_{\mathbb{R}} \in \mathcal{S}_{\{\widehat{\boldsymbol{A}}\}}^{\{\boldsymbol{M}_{+1}\}}(\mathbb{R})$$

is continuous, also at the level of Banach spaces with a uniform scaling of the value h.
Moreover, if $\boldsymbol{M}$ *satisfies also* (sm), *then* $(fG_0)|_{\mathbb{R}} \in \mathcal{F}(\mathcal{S}_{\{\boldsymbol{M}\}}^{\{\widehat{\boldsymbol{A}}\}}(0,\infty))$.

The Laplace transform on $C_{\{M\}}(0,\infty)$

The Laplace transform of $\varphi \in C_{\{M\}}(0,\infty)$ is defined as

$$\mathcal{L}(\varphi)(\zeta) = \int_0^\infty \varphi(x) e^{ix\zeta} \mathrm{d}x, \qquad \zeta \in \overline{\mathbb{H}}.$$

Lemma (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

Let M be a weight sequence satisfying (sm), and $H > 1$ be the constant appearing in (sm). Then, for every $h > 0$ the mapping $\mathcal{L}\colon C_{M,h}(0,\infty) \to \mathcal{A}_{M_{+1},Hh}(\mathbb{H})$ is well-defined and continuous. Moreover, $\mathcal{L} : C_{\{M\}}(0,\infty) \to \mathcal{A}_{\{M_{+1}\}}(\mathbb{H})$ is injective.

Injectivity of the Stieltjes moment mapping

Theorem (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

Let $\boldsymbol{M}$ be a weight sequence satisfying (sm) and $\liminf_{p\to\infty}(M_p/p!)^{1/p}>0$, and let $\boldsymbol{A}$ be a sequence satisfying (nq) and such that $\widehat{\boldsymbol{A}}$ is a weight sequence. Then, the following statements are equivalent:

(i) $\displaystyle\sum_{p=0}^{\infty}\frac{1}{m_p^{1/2}}=\infty$.

(ii) $\mathcal{B}:\mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})\to\Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

(iii) $\mathcal{M}:C_{\{\boldsymbol{M}\}}(0,\infty)\to\Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

(iv) $\mathcal{M}:\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)\to\Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

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(v) $\mathcal{M}:\mathcal{S}_{\{\boldsymbol{M}\}}^{\{\widehat{\boldsymbol{A}}\}}(0,\infty)\to\Lambda_{\{\boldsymbol{M}_{+1}\}}$ *is injective.*

$(i)\Leftrightarrow(ii)$: A result of B. Rodríguez Salinas (1955).

$(iii)\Rightarrow(iv)\Rightarrow(v)$: By restriction.

Sketch of the proof

PROOF:
$(ii) \Rightarrow (iii)$: Let $\varphi \in C_{\{M\}}(0,\infty)$ be such that $\mu_p(\varphi) = 0$ for all $p \in \mathbb{N}_0$.
Then $\mathcal{L}(\varphi) \in \mathcal{A}_{\{M_{+1}\}}(\mathbb{H})$ and $\mathcal{L}(\varphi)^{(p)}(0) = i^p \mu_p(\varphi) = 0$ for all $p \in \mathbb{N}_0$.
By assumption, $\mathcal{L}(\varphi) \equiv 0$.
Since $\mathcal{L}$ is injective, $\varphi \equiv 0$.

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$(v) \Rightarrow (ii)$: Let $f \in \mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ be such that $f^{(p)}(0) = 0$ for all $p \in \mathbb{N}_0$.
For the auxiliary function G_0, we have that $(fG_0)|_{\mathbb{R}} = \widehat{\varphi}$ for some $\varphi \in \mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty)$.
For every $p \in \mathbb{N}_0$,

$$\mu_p(\varphi) = (-i)^p \widehat{\varphi}^{(p)}(0) = (-i)^p (fG_0)^{(p)}(0) = 0.$$

By assumption, $\varphi \equiv 0$ and, thus, $fG_0 \equiv 0$. Since G_0 does not vanish, we obtain that $f \equiv 0$.

Existence of right inverses for the Stieltjes moment mapping

Theorem (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

Let $\boldsymbol{M}$ be a weight sequence satisfying (sm), and let $\boldsymbol{A}$ be a sequence satisfying (nq) and such that $\widehat{\boldsymbol{A}}$ is a weight sequence. Then:

(I) (a) Each of the following statements implies the next one:

- (i) *There exists $a>0$ such that for every $h\geq 1$ there exists a linear and continuous operator $R_h\colon \Lambda_{\boldsymbol{M}_{+1},h}\to \mathcal{S}^{\widehat{\boldsymbol{A}},ah}_{\boldsymbol{M},ah}(0,\infty)$ such that $\mathcal{M}\circ R_h$ is the identity map in $\Lambda_{\boldsymbol{M}_{+1},h}$ (i. e., $\mathcal{M}\colon \mathcal{S}^{\{\widehat{\boldsymbol{A}}\}}_{\{\boldsymbol{M}\}}(0,\infty)\to\Lambda_{\{\boldsymbol{M}_{+1}\}}$ admits local right inverses with a uniform scaling of h).*
- (ii) *$\mathcal{M}$ admits local right inverses in $\mathcal{S}_{\{\boldsymbol{M}\}}(0,\infty)$ with a uniform scaling of h.*
- (iii) *$\mathcal{M}$ admits local right inverses in $C_{\{\boldsymbol{M}\}}(0,\infty)$ with a uniform scaling of h.*
- (iv) *$\mathcal{B}$ admits local right inverses in $\mathcal{A}_{\{\boldsymbol{M}_{+1}\}}(\mathbb{H})$ with a uniform scaling of h.*

Surjectivity of the Stieltjes moment mapping

Theorem (J. Jiménez-Garrido, I. Miguel-Cantero, J. S., G. Schindl (2025))

(I) (b) The following statements are equivalent:

(i') $\mathcal{M}:\mathcal{S}_{\{M\}}^{\{\widehat{A}\}}(0,\infty)\to\Lambda_{\{M_{+1}\}}$ *is surjective.*

(ii') $\mathcal{M}:\mathcal{S}_{\{M\}}(0,\infty)\to\Lambda_{\{M_{+1}\}}$ *is surjective.*

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(v') $\gamma(\boldsymbol{M}) > 2.$

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(II) If $\boldsymbol{M}$ *satisfies* (dc), *then* (iv') *implies* (v') *(A. Debrouwere (2020)).*
So, the six conditions $(i') - (v')$ *and* (iv) *are equivalent ((iv) NEW).*

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Any of the statements $(i)-(iv)$ *implies all of* $(i')-(iv')$.
Moreover, the condition

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So, the six conditions $(i')-(v')$ *and* (iv) *are equivalent (*(iv) *NEW).*
(III) If $\mathbf{M}$ *and* $\mathbf{A}$ *satisfy* (dc)*, then all of* $(i)-(iv)$, $(i')-(v')$ *are equivalent (*$(i)-(iv)$ *NEW).*

Examples

(1) For $q > 1$ and $\sigma > 1$, $\boldsymbol{M}_{q,\sigma} := (q^{n^\sigma})_{n\in\mathbb{N}_0}$.
$\boldsymbol{M}_{q,\sigma}$ satisfies (dc) if and only if $\sigma \leq 2$; so, former results did not cover the case $\sigma > 2$.
As (sm) is satisfied for such σ and $\gamma(\boldsymbol{M}_{q,\sigma}) = \infty$, our results prove the moment mapping to be surjective onto $\Lambda_{\boldsymbol{M}_{+1}}$.

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$\boldsymbol{M}^{\tau,\sigma} = (n^{\tau n^\sigma})_{n\in\mathbb{N}_0}$ satisfies (dc) if and only if $1 < \sigma < 2$.
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(3) Our results do not apply to rapidly growing sequences not satisfying (sm).
For example, if $\boldsymbol{M} = (q^{p^p})_p$ for $q > 1$, we have $\gamma(\boldsymbol{M}) = \infty > 0$, and since (sm) fails, we know $\mathcal{M}(C_{\{\boldsymbol{M}\}}(0,\infty)) \not\subset \Lambda_{\{\boldsymbol{M}_{+1}\}}$.
Open problem: Which is the correct target space for the moment problem in such cases?

THANK YOU VERY MUCH FOR YOUR ATTENTION!

HAPPY BIRTHDAY, PEPE!!!