Generalized upper and lower Legendre conjugates for weight functions

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Introduction & motivation - I

(∗) In the field of Functional Analysis several weighted spaces are studied, e.g.:

(∗) Classes of ultradifferentiable and ultraholomorphic functions, weighted spaces of analytic functions (on $\mathbb{C}$ or $\mathbb{D}$), weighted spaces of globally defined functions of Gelfand-Shilov-type, weighted spaces of sequences of complex numbers, ...

(∗) Usually, weights are expressed in terms of a weight sequence $\mathbf{M} = (M_p)_{p\in\mathbb{N}} \in \mathbb{R}_{>0}^{\mathbb{N}}$ or in terms of a weight function $\omega : [0, +\infty) \to [0, +\infty)$.

Introduction & motivation - II

(∗) (Similar!) growth and regularity assumptions on $\mathbf{M}$ and ω are required (and unavoidable).

(∗) Conditions on weights imply, or even characterize, (desired) properties for the corresponding weighted classes.

(∗) Given $\mathbf{M}$, then one can associate the (weight) function $\omega_{\mathbf{M}}$ (S. Mandelbrojt).

(∗) Conversely, given ω, then one can associate a one-parameter family of sequences $\mathbf{W}^{(\ell)}$, $\ell > 0$; i.e. a weight matrix (A. Rainer & G.S.).

Introduction & motivation - III

(∗) Modifications of sequences appear naturally: construct new sequences out of given ones.

(∗) This gives a controlled transformation/modification of regularity measured in terms of weight sequences.

(∗) A well understood operation: Given $\mathbf{M}$, $\mathbf{N}$, then consider the convolved sequence $\mathbf{M} \star \mathbf{N}$ (H. Komatsu).

(∗) Another natural modification: multiplying and dividing sequences point-/component-wise.

Introduction & motivation - IV

(∗) (Most) prominent: $\mathbf{M} \mapsto \widehat{\mathbf{M}} := \mathbf{G}^1 \cdot \mathbf{M}$, $\mathbf{M} \mapsto \check{\mathbf{M}} := \frac{\mathbf{M}}{\mathbf{G}^1}$.

(∗) $\mathbf{G}^s := (p!^s)_{p\in\mathbb{N}}$ shall denote the Gevrey-sequence with index $s > 0$, and so $\mathbf{G}^1 \cdot \mathbf{M} = (p!M_p)_{p\in\mathbb{N}}$, $\frac{\mathbf{M}}{\mathbf{G}^1} = \left(\frac{M_p}{p!}\right)_{p\in\mathbb{N}}$.

(∗) Note: $\mathbf{M} \mapsto \widehat{\mathbf{M}}$ preserves or even increases regularity properties for $\mathbf{M}$, whereas $\check{\mathbf{M}}$ might be "non-standard" even $\mathbf{M}$ behaves "nice": consider e.g. $\mathbf{M} \equiv \mathbf{G}^s$, $1 < s < 2$.

Questions (and goals) - I

(∗) How are $\omega_{\mathbf{M}}$, $\omega_{\widehat{\mathbf{M}}}$, $\omega_{\check{\mathbf{M}}}$ related? So, how are these operations effecting the associated (weight) function $\omega_{\mathbf{M}}$?

(∗) Replace $\mathbf{G}^1$ by an arbitrary sequence $\mathbf{N}$: Consider $\mathbf{M}\cdot\mathbf{N} := (M_pN_p)_{p\in\mathbb{N}}$, $\frac{\mathbf{M}}{\mathbf{N}} := \left(\frac{M_p}{N_p}\right)_{p\in\mathbb{N}}$ and study the relation between $\omega_{\mathbf{M}}$, $\omega_{\mathbf{N}}$ and $\omega_{\mathbf{M}\cdot\mathbf{N}}$, $\omega_{\frac{\mathbf{M}}{\mathbf{N}}}$.

(∗) Again the situation is well-understood for the convolution: $\omega_{\mathbf{M}\star\mathbf{N}} = \omega_{\mathbf{M}} + \omega_{\mathbf{N}}$.

Questions (and goals) - II

(∗) Introduce operations between abstractly weight functions σ and τ yielding the analogous effect as multiplying/dividing sequences for the associated weight functions.

(∗) Work in a general setting and find applications for this approach.

(∗) Note: In general, $\frac{\mathbf{M}}{\mathbf{N}}$ can behave very irregular (oscillation) even if both $\mathbf{M}$ and $\mathbf{N}$ satisfy many (strong) growth conditions.

Known information

(∗) $\omega_{\widehat{\mathbf{M}}}$ corresponds to the lower Legendre conjugate/envelope of $\omega_{\mathbf{M}}$ and $\omega_{\check{\mathbf{M}}}$ corresponds to the upper Legendre conjugate/envelope.

(∗) These conjugates have been used for abstractly given weight functions as well; e.g. for ultradifferentiable almost analytic extensions (H.-J. Petzsche & D. Vogt, 1984) and for extension results in the ultraholomorphic setting (J. Jiménez-Garrido, J. Sanz, and G. S., 2019).

(∗) Replacing $\mathbf{G}^1$ by some/any other $\mathbf{G}^s$, $s > 0$, yields a power substitution in these conjugates.

(∗) J. Boman (2000) has given an expression relating $\omega_{\mathbf{M}}$, $\omega_{\mathbf{N}}$ and $\omega_{\mathbf{M}\cdot\mathbf{N}}$. It seems that this proof contains a formal gap: $\lim_{p\to+\infty}(M_p)^{1/p} = +\infty = \lim_{p\to+\infty}(N_p)^{1/p}$ is not clear.

Weight functions - I

We consider weight functions in the sense of J. Jiménez-Garrido, J. Sanz, and G. S. (2019):

Definition

$\omega : [0, +\infty) \to [0, +\infty)$ is called a weight function if

(∗) ω is non-decreasing and

(∗) $\lim_{t \to +\infty} \omega(t) = +\infty$.

Our definition encompasses, in particular, weight functions in the sense of Braun-Meise-Taylor.

Weight functions - II

Some notation:

(∗) Let σ, τ be weight functions. Write $\sigma \preceq \tau$ if

$$\tau(t) = O(\sigma(t)) \text{ as } t \to +\infty.$$

(∗) σ and τ are called equivalent, denoted by $\sigma \sim \tau$, if $\sigma \preceq \tau$ and $\tau \preceq \sigma$.

(∗) Write $\omega^{\iota}(t) := \omega(\frac{1}{t})$, $t > 0$.

(∗) Write $\omega^{1/\alpha}(t) := \omega(t^{1/\alpha})$, $\alpha > 0$ and $t \geq 0$.

(∗) Write $\mathrm{id}^{1/\alpha}$ for $t \mapsto t^{1/\alpha}$, $\alpha > 0$ arbitrary.

(∗) It is known: $\omega_{\mathbf{G}^s} \sim \mathrm{id}^{1/s}$.

Generalized lower Legendre conjugate

Definition

Let σ, τ be given weight functions and put

$$\sigma \check{\star} \tau(t) := \inf_{s>0}\{\sigma(s) + \tau(t/s)\}, \quad t \in [0, +\infty).$$

This generalizes the lower Legendre conjugate/envelope $h_\star(t) := \inf_{u>0}\{h(u) + tu\}$ as follows:

$$\sigma \check{\star}\, \mathrm{id}(t) = \inf_{s>0}\{\sigma(s) + t/s\} = \inf_{u>0}\{\sigma(1/u) + tu\} =: (\sigma^\iota)_\star(t).$$

More generally,

$$\forall\, \alpha > 0\ \forall\, t \geq 0 : \quad \sigma \check{\star}\, \mathrm{id}^{1/\alpha}(t) = (((\sigma^\iota)^\alpha)_\star)^{1/\alpha}(t).$$

Generalized lower Legendre conjugate - properties

Lemma

Let σ, σ_1, τ, τ_1 be weight functions.

(a) $\sigma, \tau \preceq \sigma \check{\star} \tau$;

(b) $\sigma \check{\star} \tau$ *is a weight function;*

(c) $\check{\star}$ *is commutative.*

(d) *If* $\sigma \preceq \sigma_1$ *and* $\tau \preceq \tau_1$, *then* $\sigma \check{\star} \tau \preceq \sigma_1 \check{\star} \tau_1$.

(∗) J. Boman called $\omega_{\mathbf{M}} \check{\star} \omega_{\mathbf{N}}$ "infimal convolution".

(∗) $\int \leftrightarrow$ inf and $\check{\star}$ can be defined in a wider context: $\sigma, \tau : (G_1, \odot_1) \to (G_2, \odot_2)$, then

$$\text{“}\sigma \check{\star} \tau(t) := \inf_{u,s : u \odot_1 s = t} \{\sigma(u) \odot_2 \tau(s)\}.\text{”}$$

Generalized upper Legendre conjugate

Definition

Let σ, τ be given weight functions and put

$$\sigma\widehat{\star}\tau(t) := \sup_{s\geq 0}\{\sigma(s) - \tau(s/t)\}, \quad t \in (0, +\infty).$$

A special case is (again):

$$\sigma\widehat{\star}\mathrm{id}(t) = \sup_{s\geq 0}\{\sigma(s) - s/t\} =: \sigma^{\star}(1/t) = (\sigma^{\star})^{\iota}(t), \quad t \in (0, +\infty),$$

with $\sigma^{\star}(t) := \sup_{s\geq 0}\{\sigma(s) - ts\}$ denoting the upper Legendre conjugate/envelope. More generally,

$$\forall\ \alpha > 0\ \forall\ t \geq 0: \quad \sigma\widehat{\star}\mathrm{id}^{1/\alpha}(t) = (((\sigma^{\alpha})^{\star})^{\iota})^{1/\alpha}(t).$$

Generalized upper Legendre conjugate - properties

Lemma

Let σ, τ be weight functions. Then

(a) $\sigma\widehat{\star}\tau$ *is non-decreasing and* $\lim_{t\to+\infty}\sigma\widehat{\star}\tau(t)=+\infty$.

(b) $\sigma\widehat{\star}\tau\preceq\sigma$

(c) $\lim_{t\to 0}\sigma\widehat{\star}\tau(t)=\lim_{s\to 0}\sigma(s)-\tau(s)=\sigma(0)-\tau(0)$ *and put* $\sigma\widehat{\star}\tau(0):=\sigma(0)-\tau(0)$.

But $\sigma\widehat{\star}\tau$ is in general not a weight function: $0\le\sigma\widehat{\star}\tau(t)<+\infty$ for some/any $t\in[0,+\infty)$ is not clear.

To ensure $0\le\sigma\widehat{\star}\tau(t)$ for all t it suffices to assume $\tau(0)=0$.

On the well-definedness of the upper conjugate - I

Lemma

Let σ and τ be weight functions and $t_0 \in (0, +\infty]$. Consider the following assertions:

(*i*)

$$\sup_{0<t<t_0} \limsup_{u\to+\infty} \frac{\sigma(tu)}{\tau(u)} < 1.$$

(*ii*) $\sigma \widehat{\star} \tau(t) < +\infty$ *holds for all* $t \in [0, t_0)$; *i.e.*

$$\forall\, t \in (0, t_0)\ \exists\ D_t > 0\ \forall\ s \geq 0 : \quad \sigma(s) - \tau(s/t) \leq D_t. \quad (1)$$

(*iii*)

$$\sup_{0<t<t_0} \limsup_{u\to+\infty} \frac{\sigma(tu)}{\tau(u)} \leq 1.$$

Then (i) ⇒ (ii) ⇒ (iii) is valid.

On the well-definedness of the upper conjugate - II

In order to ensure that $\sigma\widehat{\star}\tau$ is a weight function one shall assume that

(A) $\tau(0) = 0$ and

(B) (1) holds with $t_0 = +\infty$ and so $\sigma\widehat{\star}\tau(t) < +\infty$ for any $t > 0$.

If $\sigma\widehat{\star}\sigma$ is well-defined, then

$$\forall\ t > 0: \quad \lim_{u\to+\infty} \frac{\sigma(tu)}{\sigma(u)} = 1,$$

which precisely means that σ is slowly varying.

On the well-definedness of the upper conjugate - III

Proposition

Let σ be a weight function and $\alpha > 0$. Then the following are equivalent:

(i) $\sigma \widehat{\star} \mathrm{id}^{1/\alpha}(t) < +\infty$ *for all* $t \in (0, +\infty)$.

(ii) $\sigma(s) = o(s^{1/\alpha})$ *as* $s \to +\infty$.

Note: $\mathrm{id}^{1/\alpha}(0) = 0$ is clear.

More generally, one can show:

On the well-definedness of the upper conjugate - IV

Proposition

Let σ, τ be weight functions.

(*i*) *If either σ or τ satisfies*

$$\omega(2t) = O(\omega(t)), \quad t \to +\infty,$$

then $\sigma(s) = o(\tau(s))$ as $s \to +\infty$ implies $\sigma \widehat{\star} \tau(t) < +\infty$ for all $t \in (0, +\infty)$.

(*ii*) *If either σ or τ satisfies*

$$\exists\, H \geq 1\ \forall\, t \geq 0 : \quad 2\omega(t) \leq \omega(Ht) + H,$$

then $\sigma \widehat{\star} \tau(t) < +\infty$ for all $t \in (0, +\infty)$ implies $\sigma(s) = o(\tau(s))$ as $s \to +\infty$.

Limiting example

Example

Consider the weight function $\log_+$:

$$\log_+(t) := 0, \quad t \in [0,1), \qquad \log_+(t) := \log(t), \quad t \geq 1.$$

Then

$$\log_+ \check{\star} \log_+ = \log_+, \qquad \log_+ \widehat{\star} \log_+ = \log_+ .$$

Note: $\log_+$ corresponds to $\omega_{\mathbf{M}}$ when $\mathbf{M}$ is such that $M_p = +\infty$ for all sufficiently large p - so formally not a weight sequence (G.S., 2020).

Growth indices (J. Jiménez-Garrido, J. Sanz, G. S. (2019))

Let $\omega : [0, +\infty) \to [0, +\infty)$ be a weight function and $\gamma > 0$.

- (∗) ω has $(P_{\omega,\gamma})$ if

$$\exists\, K > 1 : \quad \limsup_{t \to +\infty} \frac{\omega(K^\gamma t)}{\omega(t)} < K.$$

- (∗) Then put

$$\underline{\gamma}(\omega) := \sup\{\gamma > 0 : \ (P_{\omega,\gamma}) \text{ is satisfied}\}.$$

- (∗) Analogously, ω has $(\overline{P}_{\omega,\gamma})$ if

$$\exists\, A > 1 : \quad \liminf_{t \to +\infty} \frac{\omega(A^\gamma t)}{\omega(t)} > A.$$

- (∗) Then put

$$\overline{\gamma}(\omega) := \inf\{\gamma > 0 : \ (\overline{P}_{\omega,\gamma}) \text{ is satisfied}\}.$$

- (∗) $(0 \le)\underline{\gamma}(\omega) \le \overline{\gamma}(\omega)$ holds.

$\check{\star}$ and growth indices

Theorem

Let σ, τ be weight functions.

(i) *If $\gamma(\sigma), \gamma(\tau) > 0$, then*

$$\gamma(\sigma) + \gamma(\tau) \leq \gamma(\sigma \check{\star} \tau).$$

(ii) *If $\overline{\gamma}(\sigma), \overline{\gamma}(\tau) < +\infty$, then*

$$\overline{\gamma}(\sigma \check{\star} \tau) \leq \overline{\gamma}(\sigma) + \overline{\gamma}(\tau).$$

$\widehat{\star}$ and growth indices

Theorem

Let σ, τ be weight functions such that $\tau(0) = 0$ and $\sigma\widehat{\star}\tau$ is well-defined.

(i) *If $\gamma(\sigma) > 0$ and $\overline{\gamma}(\tau) < +\infty$, then*

$$\gamma(\sigma) \leq \gamma(\sigma\widehat{\star}\tau) + \overline{\gamma}(\tau).$$

(ii) *If $0 < \gamma(\tau) \leq \overline{\gamma}(\sigma) < +\infty$, then*

$$\overline{\gamma}(\sigma\widehat{\star}\tau) + \gamma(\tau) \leq \overline{\gamma}(\sigma).$$

Basic notation and conditions - I

Let $\mathbf{M} = (M_p)_{p\in\mathbb{N}} \in \mathbb{R}_{>0}^{\mathbb{N}}$.

(∗) $\mathbf{M}$ is log-convex (H. Komatsu's condition $(M.1)$), if

$$\forall\, p \in \mathbb{N}_{>0}: \quad M_p^2 \leq M_{p-1} M_{p+1}.$$

(∗) Set

$$\mathbf{M}_\iota := \liminf_{p\to+\infty}\left(\frac{M_p}{M_0}\right)^{1/p} = \liminf_{p\to+\infty}(M_p)^{1/p}.$$

(∗) If $\mathbf{M}$ is log-convex, then

$$\lim_{p\to+\infty}\left(\frac{M_p}{M_0}\right)^{1/p} = \mathbf{M}_\iota \in (0,+\infty].$$

(∗) In this situation, the case $\mathbf{M}_\iota \neq +\infty$ is non-standard.

Basic notation and conditions - II

(∗) Write $\mathbf{M} \preceq \mathbf{N}$ if

$$\sup_{p\in\mathbb{N}_{>0}} \left(\frac{M_p}{N_p}\right)^{1/p} < +\infty.$$

(∗) Write $\mathbf{M} \lhd \mathbf{N}$ if

$$\lim_{p\to+\infty} \left(\frac{M_p}{N_p}\right)^{1/p} = 0.$$

Associated (weight) function

Definition

The associated function $\omega_{\mathbf{M}} : [0,+\infty) \to [0,+\infty) \cup \{+\infty\}$ is given as follows (use the convention $0^0 := 1$):

$$\omega_{\mathbf{M}}(t) := \sup_{p\in\mathbb{N}} \log\left(\frac{M_0 t^p}{M_p}\right), \qquad t \geq 0.$$

Lemma

Let $\mathbf{M} \in \mathbb{R}_{>0}^{\mathbb{N}}$ *be given.*

(*i*) $\mathbf{M}_\iota < +\infty$ *implies*

$$\forall\ t > \mathbf{M}_\iota : \quad \omega_{\mathbf{M}}(t) = +\infty.$$

(*ii*) *If* $\lim_{p\to+\infty}(M_p)^{1/p} = +\infty$, *then* $\omega_{\mathbf{M}}(t) < +\infty$ *for all* $t \geq 0$.

Characterization of well-definedness

Proposition

Let $\mathbf{M}, \mathbf{N} \in \mathbb{R}_{>0}^{\mathbb{N}}$, *assume that* $\mathbf{N}$ *is log-convex, so* $\mathbf{N}_\iota \in (0, +\infty]$. *The following are equivalent:*

(*i*)

$$\forall\, t \geq 0\ \exists\, D_t \geq 1\ \forall\, u \in [0, \mathbf{N}_\iota) : \quad \omega_{\mathbf{M}}(tu) \leq \omega_{\mathbf{N}}(u) + D_t.$$

(*ii*) $\mathbf{N} \lhd \mathbf{M}$ *holds.*

Corollary

Let $\mathbf{M} \in \mathbb{R}_{>0}^{\mathbb{N}}$ *be log-convex and such that* $\mathbf{M}_\iota = +\infty$. *Then* $\omega_{\mathbf{M}} \widehat{\star} \omega_{\mathbf{M}}$ *is not well-defined.*

Main theorem for the lower conjugate

Theorem

Let $\mathbf{M}, \mathbf{N}$ *be log-convex. Then*

$$\forall\, t \in [0, \mathbf{M}_\iota \cdot \mathbf{N}_\iota) : \quad \omega_{\mathbf{M}\cdot\mathbf{N}}(t) = \omega_{\mathbf{M}} \check{\star} \omega_{\mathbf{N}}(t),$$

with the natural convention $\mathbf{M}_\iota \cdot \mathbf{N}_\iota = +\infty$ *if either* $\mathbf{M}_\iota = +\infty$ *or* $\mathbf{N}_\iota = +\infty$.

Consequence (recall the formula for the convolution):

(∗) "Easy modifications for weight sequences correspond to complicated modifications for (associated) weight functions"

(∗) "Complicated modifications for weight sequences correspond to easy modifications for (associated) weight functions"

Main theorem for the upper conjugate

Theorem

Let log-convex sequences $\mathbf{M}, \mathbf{N}$ *be given such that*

(a) $\mathbf{M}_\iota = +\infty = \mathbf{N}_\iota$,

(b) $\mathbf{N} \lhd \mathbf{M}$.

Then

$$\forall\ t \in [0, +\infty) : \quad \omega_{\mathbf{M}} \widehat{\star} \omega_{\mathbf{N}}(t) \leq \omega_{\frac{\mathbf{M}}{\mathbf{N}}}(t)$$

and, if $\frac{\mathbf{M}}{\mathbf{N}}$ *is even log-convex, then equality holds for all* $t \in [0, +\infty) = [0, \frac{\mathbf{M}}{\mathbf{N}}_\iota)$.

If (b) *is replaced by the weaker relation*

(c) $\mathbf{N} \preceq \mathbf{M}$,

then one has to restrict to $t \in [0, \frac{\mathbf{M}}{\mathbf{N}}_\iota)$.

Upper vs. lower conjugate - Main result

Theorem

Let log-convex sequences $\mathbf{M}, \mathbf{N}$ *be given.*

(*i*) *If* $\mathbf{N}_\iota = +\infty$, *then*

$$\forall\ t \in [0, \mathbf{M}_\iota) : \quad (\omega_{\mathbf{M}} \check{\star} \omega_{\mathbf{N}}) \widehat{\star} \omega_{\mathbf{N}}(t) = \omega_{\mathbf{M}}(t),$$

and if $\mathbf{M}_\iota = +\infty$, *then*

$$\forall\ t \in [0, \mathbf{N}_\iota) : \quad (\omega_{\mathbf{M}} \check{\star} \omega_{\mathbf{N}}) \widehat{\star} \omega_{\mathbf{M}}(t) = \omega_{\mathbf{N}}(t).$$

(*ii*) *Assume that* $\mathbf{M}_\iota = +\infty$, $\mathbf{N} \preceq \mathbf{M}$, *and that* $\frac{\mathbf{M}}{\mathbf{N}}$ *is log-convex. Then we obtain*

$$\forall\ t \in [0, +\infty) : \ \omega_{\mathbf{M}}(t) = \omega_{\mathbf{N}} \check{\star} (\omega_{\mathbf{M}} \widehat{\star} \omega_{\mathbf{N}})(t) = (\omega_{\mathbf{M}} \widehat{\star} \omega_{\mathbf{N}}) \check{\star} \omega_{\mathbf{N}}(t).$$

Definitions and notions - I

(a) Let $\omega : [0, +\infty) \to [0, +\infty)$ be continuous, non-decreasing, $\omega(t) = 0$ for all $t \in [0, 1]$.

(b) $\log(t) = o(\omega(t))$ as $t \to +\infty$.

(c) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

Introduce the set (of basic weight functions in sense of Braun-Meise-Taylor):

$$\mathcal{W}_0 := \{\omega : [0, \infty) \to [0, \infty) : \omega \text{ satisfies } (a), (b), (c)\}.$$

Let $\omega \in \mathcal{W}_0$ and consider the Legendre-Fenchel-Young-conjugate

$$\varphi^*_\omega(x) := \sup\{xy - \varphi_\omega(y) : y \geq 0\}, \quad x \geq 0.$$

Definitions and notions - II

(∗) Associate the weight matrix $\mathcal{M}_\omega := \{\mathbf{W}^{(\ell)} : \ell > 0\}$ by

$$W_p^{(\ell)} := \exp\left(\frac{1}{\ell}\varphi_\omega^*(\ell p)\right).$$

(∗) Properties for ω transfer to $\mathcal{M}_\omega$ - and vice versa via exploiting (c).

(∗) If in addition

$$(\omega_1): \quad \omega(2t) = O(\omega(t)), \quad t \to +\infty,$$

then weighted spaces defined via ω can be described equivalently via $\mathcal{M}_\omega$.

(∗) $\gamma(\omega) > 0$ if and only if (ω_1) holds.

(∗) Indeed, for considering φ_ω^*, defining $\mathcal{M}_\omega$ and the corresponding spaces, (c) is not necessary.

Products and quotients of weight matrices

Let $\sigma, \tau \in \mathcal{W}_0$ be given with associated matrices $\mathcal{M}_\sigma$, $\mathcal{M}_\tau$ and $\mathbf{N} \in \mathbb{R}_{>0}^{\mathbb{N}}$. Introduce:

(∗)
$$\mathcal{M}_\sigma \cdot \mathcal{M}_\tau := \{\mathbf{S}^{(\ell)} \cdot \mathbf{T}^{(\ell)} : \ell > 0\},$$

(∗)
$$\mathcal{M}_\sigma \cdot \mathbf{N} := \{\mathbf{S}^{(\ell)} \cdot \mathbf{N} : \ell > 0\},$$

(∗)
$$\frac{\mathcal{M}_\sigma}{\mathcal{M}_\tau} := \left\{\frac{\mathbf{S}^{(\ell)}}{\mathbf{T}^{(\ell)}} : \ell > 0\right\},$$

(∗)
$$\frac{\mathcal{M}_\sigma}{\mathbf{N}} := \left\{\frac{\mathbf{S}^{(\ell)}}{\mathbf{N}} : \ell > 0\right\}.$$

First main result(s) - lower conjugate

Theorem

Let $\sigma, \tau \in \mathcal{W}_0$ be given with associated matrices $\mathcal{M}_\sigma$, $\mathcal{M}_\tau$. If either σ or τ satisfies in addition (ω_1), then as l.c.v.s.

$$\forall\, \ell, j > 0 : \quad \mathfrak{F}_{[\mathcal{M}_\sigma \cdot \mathcal{M}_\tau]} = \mathfrak{F}_{[\omega_{\mathbf{S}^{(\ell)} \cdot \mathbf{T}^{(j)}}]} = \mathfrak{F}_{[\sigma \check{\star} \tau]}.$$

Corollary

Let $\sigma \in \mathcal{W}_0$ be given with associated matrix $\mathcal{M}_\sigma$ and let $\alpha > 0$. Then, for all $\ell > 0$ as l.c.v.s. we get

$$\mathfrak{F}_{[\mathcal{M}_\sigma \cdot \mathbf{G}^\alpha]} = \mathfrak{F}_{[\omega_{\mathbf{S}^{(\ell)} \cdot \mathbf{G}^\alpha}]} = \mathfrak{F}_{[\omega_{\mathbf{S}^{(\ell)}} \check{\star} \omega_{\mathbf{G}^\alpha}]} = \mathfrak{F}_{[\sigma \check{\star} \omega_{\mathbf{G}^\alpha}]}$$
$$= \mathfrak{F}_{[\sigma \check{\star}\, \mathrm{id}^{1/\alpha}]} = \mathfrak{F}_{[(((\sigma^\iota)^\alpha)_\star)^{1/\alpha}]}.$$

(∗) Here, $\mathfrak{F} \in \{\mathcal{E}, \mathcal{B}, \mathcal{A}, \mathcal{S}, \Lambda, \mathcal{F}\}$ (weighted categories).

(∗) $[\cdot]$ is a uniform notation meaning either the Roumieu case $\{\cdot\}$ or the Beurling case $(\cdot)$.

(∗) For any $\omega \in \mathcal{W}_0$ consider the Fréchet space

$$\mathcal{S}_{(\omega)}(\mathbb{R}) :=$$

$$\{f \in \mathcal{E}(\mathbb{R}) : \sup_{x\in\mathbb{R}, j,k\in\mathbb{N}} \frac{(1+|x|)^k |f^{(j)}(x)|}{W^{(\ell)}_{j+k}} < +\infty, \ \forall\, \ell > 0\}.$$

(∗) For the corresponding associated matrix $\mathcal{M}_\omega$ set

$$\mathcal{S}_{(\mathcal{M}_\omega)}(\mathbb{R}) :=$$

$$\{f \in \mathcal{E}(\mathbb{R}) : \sup_{x\in\mathbb{R}, j,k\in\mathbb{N}} \frac{(1+|x|)^k |f^{(j)}(x)|}{h^{j+k} W^{(\ell)}_{j+k}} < +\infty, \ \forall\, \ell > 0 \ \forall\, h > 0\}.$$

Second main result - upper conjugate

Theorem

Let $\omega \in \mathcal{W}_0$ be given and $\alpha > 0$. Assume:

(i) *There exists $\ell_0 > 0$ such that $\gamma(\mathbf{W}^{(\ell_0)}) > \alpha$ (growth index by V. Thilliez for sequences).*

(ii) $\omega(t) = o(t^{1/\alpha})$ *as $t \to +\infty$.*

Then, for all $c, \ell > 0$ it follows that as l.c.v.s.

$$\mathfrak{F}_{[\frac{\mathcal{M}_\omega}{\mathbf{G}^\alpha}]} = \mathfrak{F}_{[\omega_{\mathbf{W}^{(c\ell_0)}/\mathbf{G}^\alpha}]} = \mathfrak{F}_{[\omega_{\mathbf{W}^{(\ell)}}\widehat{\star}\omega_{\mathbf{G}^\alpha}]} = \mathfrak{F}_{[\omega\widehat{\star}\omega_{\mathbf{G}^\alpha}]}$$
$$= \mathfrak{F}_{[\omega\widehat{\star}\,\mathrm{id}^{1/\alpha}]} = \mathfrak{F}_{[(((\omega^\alpha)^\star)^\iota)^{1/\alpha}]},$$

Corresponding result for $\frac{\mathcal{M}_\sigma}{\mathcal{M}_\tau}$ requires knowledge on the regularity of $\mathbf{S}^{(\ell)}/\mathbf{T}^{(\ell)}$ $\rightsquigarrow$ new growth index $\gamma(\mathcal{M}_\omega)$ resp. $\gamma(\mathcal{M})$!

$\check{\star}$ and the composition operator on Gelfand-Shilov-type classes

Let $\psi(x) := x^2 + \frac{1}{4}$, $x \in \mathbb{R}$, ψ_m denotes the m-th iteration of ψ and set $C_{\psi_m} : f \mapsto f \circ \psi_m$.

Theorem

Let $\sigma, \tau \in \mathcal{W}_0$, $0 < \alpha < 2$, $\mu \in \mathbb{C}$ with $|\mu| > 1$ and set $R_\mu := \sum_{m=0}^{+\infty} \frac{C_{\psi_m}}{\mu^{m+1}}$ (i.e. $R_\mu = (\mu - C_\psi)^{-1}$). We assume that

(a) $\gamma(\sigma) > 1$,

(b) $\mathcal{M}_\sigma$, $\mathcal{M}_\tau$ *are related by*

$$\exists\, k_0 > 0\ \forall\, \ell > 0 : \quad \mathbf{T}^{(k_0)} \preceq \mathbf{S}^{(\ell)}.$$

Then there exists $f \in \mathcal{S}_{(\sigma)}(\mathbb{R})$ such that $R_\mu(f) \notin \mathcal{S}_{((((\tau^\iota)^\alpha)_\star)^{1/\alpha})}(\mathbb{R}) = \mathcal{S}_{(\tau \check{\star} \mathrm{id}^{1/\alpha})}(\mathbb{R}) = \mathcal{S}_{(\mathcal{M}_\tau \cdot \mathbf{G}^\alpha)}(\mathbb{R})$.

$\widehat{\star}$ and the composition operator on Gelfand-Shilov-type classes

Theorem

Let $\sigma, \tau \in \mathcal{W}_0$, $0 < \alpha < 2$, $\mu \in \mathbb{C}$ *with* $|\mu| > 1$ *and* R_μ *as before. We assume:*

(a) *There exists* $\ell_0 > 0$ *such that* $\gamma(\mathbf{S}^{(\ell_0)}) > \alpha + 1$.

(b) $\sigma(t) = o(t^{1/\alpha})$ *as* $s \to +\infty$.

(c) $\mathcal{M}_\sigma$, $\mathcal{M}_\tau$ *are related by*

$$\exists\, k_0 > 0\, \forall\, \ell > 0 : \quad \mathbf{T}^{(k_0)} \preceq \mathbf{S}^{(\ell)}.$$

Then there exists $f \in \mathcal{S}_{((((\sigma^\alpha)^\star)^\iota)^{1/\alpha})}(\mathbb{R}) = \mathcal{S}_{(\sigma \widehat{\star} \mathrm{id}^{1/\alpha})}(\mathbb{R}) = \mathcal{S}_{(\frac{\mathcal{M}_\sigma}{\mathbf{G}^\alpha})}(\mathbb{R})$ *such that* $R_\mu(f) \notin \mathcal{S}_{(\tau)}(\mathbb{R})$.

Generalization of known result

These results are generalizing one of the main statements by H. Ariza, C. Fernández, and A. Galbis (2025) for Gevrey weights:

Let $d, d' > 1$ be given with $d < d' < d+2$, set $\alpha := d' - d$.

(∗) Apply the first result to $\sigma = \tau := \mathrm{id}^{1/d}$, so

$$(((\sigma^{\iota})^{\alpha})_{\star})^{1/\alpha} = \mathrm{id}^{1/d} \check{\star}\, \mathrm{id}^{1/\alpha} \sim \omega_{\mathbf{G}^d} \check{\star} \omega_{\mathbf{G}^{\alpha}} = \omega_{\mathbf{G}^{d+\alpha}} = \omega_{\mathbf{G}^{d'}}.$$

(∗) Apply the second result to $\sigma = \tau := \mathrm{id}^{1/d'}$, so

$$(((\sigma^{\alpha})^{\star})^{\iota})^{1/\alpha} = \mathrm{id}^{1/d'} \widehat{\star}\, \mathrm{id}^{1/\alpha} \sim \omega_{\mathbf{G}^{d'}} \widehat{\star} \omega_{\mathbf{G}^{\alpha}} = \omega_{\mathbf{G}^{d'-\alpha}} = \omega_{\mathbf{G}^d}.$$

(∗) H. Ariza, C. Fernández, and A. Galbis, Iterates of composition operators on global spaces of ultradifferntiable functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Mat. RACSAM 119, art. no. 9, 2025.

(∗) J. Boman, Uniqueness and non-uniqueness for microanalytic continuation of ultradistributions, Contemp. Mathem. 251, 61-82, 2000

(∗) G. Schindl, Generalized upper and lower Legendre conjugates for weight functions, 2025, available online at https://arxiv.org/pdf/2505.07497.pdf

(∗) G. Schindl, Generalized upper and lower Legendre conjugates for Braun-Meise-Taylor weight functions, 2025, available online at https://arxiv.org/pdf/2505.17725.pdf