Bergman-Toeplitz operators on periodic planar domains

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Selection of papers of JT with José Bonet

Non–distinguished Fréchet function spaces. Bull. Soc. Roy. Sci. Liége 58 (1989), 483–490.

The subspace problem for weighted inductive limits of spaces of holomorphic functions. Michigan Math.J. 42 (1995), 259–268.

(with M.Engliš) *Weighted L^∞–estimates for Bergman projections.* Studia Math. 171,1 (2005), 67–92.

Toeplitz–operators on the space of analytic functions with logarithmic growth. J. Math. Anal. Appl. 353 (2009), 428–435.

Solid hulls of weighted Banach spaces of analytic functions on the unit disc with exponential weights. Ann. Acan. Sci. Fenn. Math. 43 (2018), 521–530.

(with W.Lusky) *On boundedness and compactness of Toeplitz operators in weighted H^∞-spaces.* J. Functional Analysis 278, 10 (2020), 108456.

Abstract

• We introduce **Floquet-transform techniques** to study Bergman spaces, Bergman kernels and Toeplitz operators T_a on **unbounded periodic planar domains** Π, which are defined as the union of infinitely many copies of the translated, bounded **periodic cell** ϖ.

• The Floquet-transform yields a connection between the Bergman projection $P_\Pi : L^2(\Pi) \to A^2(\Pi)$ and a family of Bergman-type projections P_η in the space $L^2(\varpi)$, where $\eta \in [-\pi, \pi]$ is the so-called Floquet variable. We get an explicit formula for the corresponding kernels.

• We study Toeplitz operators $T_a : A^2(\Pi) \to A^2(\Pi)$ with periodic symbols. Floquet-transform establishes a connection of T with family of Toepliz-type operators $T_{a,\eta}$, $\eta \in [-\pi, \pi]$, in the cell ϖ. In particular, we prove the **"spectral band formula"**, which describes the essential spectrum of T_a in terms of the spectra of the operators $T_{a,\eta}$.

Abstract

• As an **example and application**, we consider (simply connected) periodic domains Π_h containing thin geometric structures and show how to construct a Toeplitz operator $T_a : A^2(\Pi_h) \to A^2(\Pi_h)$ such that, for any $N \in \mathbb{N}$,

The essential spectrum of T_a contains N disjoint components which approximatively coincide with any given finite set $x_1, \ldots, x_N$ of real numbers.

• Using a Riemann mapping from the disc $\mathbb{D}$ onto Π_h one can then find a Toeplitz operator $T : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ with a bounded symbol and with the same spectral properties as T_a.

J. Taskinen: On the Bergman projection and kernel in periodic planar domains. Proceedings of IWOTA 2022 Lancaster.
J. Taskinen: On Bergman-Toeplitz operators in periodic planar domains. Transactions LMS (2025).

Bergman projection, Toeplitz operator

Given a domain Ω in the complex plane $\mathbb{C}$, we denote by $L^2(\Omega)$ the Lebesgue-Hilbert space with respect to the (real) area measure dA and by $A^2(\Omega)$ the corresponding Bergman space, which is the closed subspace consisting of analytic functions.
We denote by P_Ω the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. It can always written with the help of the Bergman kernel $K_\Omega : \Omega \times \Omega \to \mathbb{C}$,

$$P_\Omega f(z) = \int\limits_\Omega K_\Omega(z,w) f(w) dA(w), \qquad z \in \Omega,\ f \in L^2(\Omega).$$

Given $a \in L^\infty(\Omega)$, the Toeplitz operator T_a with symbol a is defined by

$$T_a f(z) = P_\Omega M_a f(z) = \int\limits_\Omega K_\Omega(z,w) a(w) f(w) dA(w), \quad z \in \Omega,\ A \in L^2(\Omega).$$

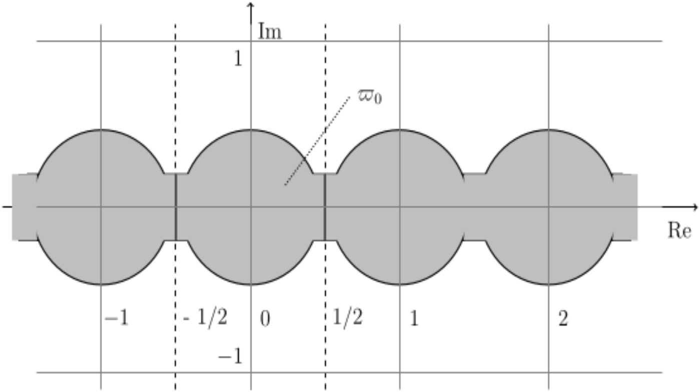
Periodic domain and cell

• Periodic cell $\varpi \subset]-\frac{1}{2},\frac{1}{2}[\times]-M,M[\subset \mathbb{R}^2 \cong \mathbb{C}$ for some $M > 0$, see picture below. Translates of ϖ are $\varpi_m = \varpi + m$, where $m \in \mathbb{Z} \subset \mathbb{C}$,
• Periodic domain Π is the interior of the set

$$\bigcup_{m\in\mathbb{Z}} \mathrm{cl}(\varpi_m).$$

• Some geometric assumptions: ϖ and Π are Lipschitz domains such that the boundaries $\partial\varpi$ and $\partial\Pi$ are in addition piecewise smooth. Excludes cusps both in ϖ and Π. Consequently, $\partial\varpi$ is a Jordan curve, polynomials form a dense subspace of the Bergman space $A^2(\varpi)$.

Periodic domain and cell



Floquet transform in $L^2(\Pi)$

The definition of the Floquet transform reads for $f \in L^2(\Pi)$ as

$$\mathsf{F}f(z,\eta) = \widehat{f}(z,\eta) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} e^{-i\eta m} f(z+m), \quad z \in \varpi, \ \eta \in [-\pi,\pi],$$

$$\mathsf{F} : L^2(\Pi) \to L^2\big(-\pi,\pi; L^2(\varpi)\big) \ .$$

Here, $L^2(-\pi,\pi;L^2(\varpi))$ is the vector valued L^2-space (or Bochner space) on $[-\pi,\pi]$ of functions $g = g(z,\eta)$ with values $g(\cdot,\eta)$ in $L^2(\varpi)$, with norm

$$\|g\|^2 = \int_{-\pi}^{\pi} \|g(\cdot,\eta)\|_{L^2(\varpi)}^2 d\eta$$

The series converges in $L^2(-\pi,\pi;L^2(\varpi))$, thus pointwise for a.e. η, z etc.

Theorem

F *is a unitary map from* $L^2(\Pi)$ *onto* $L^2(-\pi,\pi;L^2(\varpi))$ *with inverse*

$$\mathsf{F}^{-1}g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i[\mathrm{Re} z]\eta} g(z - [\mathrm{Re}\, z], \eta) d\eta, \quad z \in \Pi. \tag{1}$$

Floquet transform in $A^2(\Pi)$

The Floquet transform is simply defined in $A^2(\Pi) \subset L^2(\Pi)$ as the restriction. Then, the question is about its range.

Theorem

Floquet transform F *maps* $A^2(\Pi)$ *onto* $L^2(-\pi, \pi; A^2_\eta(\varpi))$*. Its inverse* $\mathsf{F}^{-1} : L^2(-\pi, \pi; A^2_\eta(\varpi)) \to A^2(\Pi)$ *is given by the formula (1).*

- For $\eta \in [-\pi, \pi]$, we denote by $A^2_{\eta,\mathrm{ext}}(\varpi)$ the subspace of $A^2(\varpi)$ of such f which can be extended as analytic functions to a neighborhood in Π of $\mathrm{cl}\,(\varpi) \cap \Pi$ and satisfy the boundary condition

$$f(\frac{1}{2} + iy) = e^{i\eta} f(-\frac{1}{2} + iy) \quad \text{for all } a < y < b.$$

- We define the space $A^2_\eta(\varpi)$ as the closure of $A^2_{\eta,\mathrm{ext}}(\varpi)$ in $A^2(\varpi)$.

Projections in Π and in ϖ

- We denote by $P_\eta : L^2(\varpi) \to A^2_\eta(\varpi)$ the orthogonal projection with kernel $K_\eta : \varpi \times \varpi \to \mathbb{C}$,

$$P_\eta f(z) = \int_\varpi K_\eta(z,w) f(w) dA(w).$$

Lemma

The map $\mathcal{P}f(z,\eta) = \big(P_\eta f(\cdot,\eta)\big)$ *is the orthogonal projection from* $L^2(-\pi,\pi; L^2(\varpi))$ *onto* $L^2(-\pi,\pi; A^2_\eta(\varpi))$.

Taking the Floquet transform yields a connection between the Bergman projections in ϖ and Π.

Theorem

The Bergman projection $P_\Pi : L^2(\Pi) \to A^2(\Pi)$ *equals* $P_\Pi = \mathsf{F}^{-1}\mathcal{P}\mathsf{F}$. *The kernel* K_Π *can be written as*

$$K_\Pi(z,w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\eta([\mathrm{Re} z]-[\mathrm{Re} w])} K_\eta(z-[\mathrm{Re} z], w-[\mathrm{Re} w])d\eta$$

Kernel in Π in terms of a conformal mapping

If the domain Π is simply connected then its Bergman kernel equals

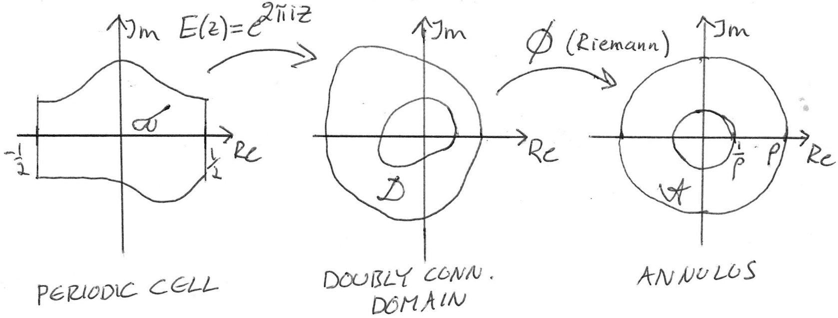
$$K_\Pi(z,w) = C_\rho^2 \pi^{-1} \widetilde{K}_\Pi(z,w)\,\mathrm{sech}^2\big(C_\rho(\varphi(z) - \overline{\varphi(w)}\big),$$

where $C_\rho = \pi^2/(2\log\rho)$ and

$$\widetilde{K}_\Pi(z,w) = e^{i2\pi(z-\varphi(z)-\bar{w}+\overline{\varphi(w)})}\phi'(e^{i2\pi z})\overline{\phi'(e^{i2\pi w})}$$

For the strip $\Sigma = (-\infty,\infty)\times(-\pi,\pi)$ we obtain (known)

$$K_\Sigma(z,w) = (16\pi)^{-1}\mathrm{sech}^2\big((z-\bar{w})/4\big).$$



Toeplitz-type operators on $A^2_\eta(\varpi)$

From now on we consider Toeplitz operators $T_a : A^2(\Pi) \to A^2(\Pi)$ with periodic symbols $a \in L^\infty(\Pi)$: we assume

$$a(z) = a(z+1) \qquad \text{for a.e. } z \in \Pi.$$

• We define for all $\eta \in [-\pi, \pi]$ the bounded, Toeplitz-type operator $T_{a,\eta} : A^2_\eta(\varpi) \to A^2_\eta(\varpi)$,

$$T_{a,\eta} f = P_\eta(a|_\varpi f)$$

• In the Bochner space, $\mathcal{T}_a : L^2(-\pi, \pi; A^2_\eta(\varpi)) \to L^2(-\pi, \pi; A^2_\eta(\varpi))$,

$$\mathcal{T}_a : f(\cdot, \eta) \mapsto T_{a,\eta} f(\cdot, \eta),$$

The following is an immediate consequence of the definitions.

Lemma

$T_a f = \mathsf{F}^{-1} \mathcal{T}_a \mathsf{F} f$ *for all* $f \in A^2(\Pi)$.

"Spectral band formula"

We denote the spectrum of $T_{a,\eta}$ in the space $A^2_\eta(\varpi)$ by $\sigma(T_{a,\eta})$.

Theorem

The essential spectrum of the Toeplitz-operator $T_a : A^2(\Pi) \to A^2(\Pi)$ can be described by the formula

$$\sigma_{\rm ess}(T_a) = \bigcup_{\eta \in [-\pi,\pi]} \sigma(T_{a,\eta}).$$

Moreover, there holds $\sigma(T_a) = \sigma_{\rm ess}(T_a)$.

(An analogous formula is classical in spectral problems for periodic elliptic operators which are in particular *self-adjoint, unbounded operators in L^2-spaces*; S.A.Nazarov, P.Kuchment and many others.)

To prove the inclusion "$\subset$" one first shows that the set $\Sigma := \cup_{\eta \in [-\pi,\pi]} \sigma(T_{a,\eta})$ is closed. Then, if $\lambda \in \mathbb{C} \setminus \Sigma$, its distance to the spectra of all $T_{a,\eta}$ is above some small positive number, and it is not difficult to construct a bounded inverse for the operator $T_a - \lambda \mathrm{Id}$. We get

$$\sigma_{\rm ess}(T_a) \subset \sigma(T_a) \subset \Sigma.$$

"Spectral band formula": on the proof.

To prove $\sigma_{\mathrm{ess}}(T_a) \supset \bigcup_{\eta \in [-\pi,\pi]} \sigma(T_{a,\eta})$ we construct a Weyl singular sequence.

Lemma

If H is Hilbert space, $T \in \mathcal{L}(H)$ and $\lambda \in \mathbb{C}$, then $\lambda \in \sigma_{\mathrm{ess}}(T)$, if and only if there exists a Weyl singular sequence, which is a sequence $(h_n)_{n=1}^{\infty} \subset H$ with no convergent subsequences such that $\|h_n\|_H = 1$ for all n, and

$$\lim_{n\to\infty} \|Th_n - \lambda h_n\|_H = 0$$

Assume $\lambda \in \Sigma = \cup_{\eta \in [-\pi,\pi]} \sigma(T_{a,\eta})$ and for example that λ is an eigenvalue of $T_{a,\nu}$ with a fixed $\nu \in [-\pi,\pi]$. Then, we pick up an eigenfunction $f \in A^2_\nu(\varpi)$, extend it suitably to an element $f \otimes \chi_n$ of $L^2(-\pi,\pi; A^2_\eta(\varpi))$ and use the inverse Floquet transform to define an element $g_n := \mathsf{F}^{-1}(f \otimes \chi_n)$ of $A^2(\Pi)$. (This is however not an eigenfunction of T_a. The functions χ_n are approximations of the Dirac measure of the point $\nu \in [-\pi,\pi]$)

The singular sequence is obtained by taking "sparse" translations of g_n.

Toeplitz operator on $\mathbb{D}$ with interesting essential spectra

We start with a (compact, self-adjoint) Toeplitz operator $T_b : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ which has a radial, real-valued, bounded symbol, compactly supported in $\mathbb{D}$. Then, T_b is the Taylor coefficient multiplier

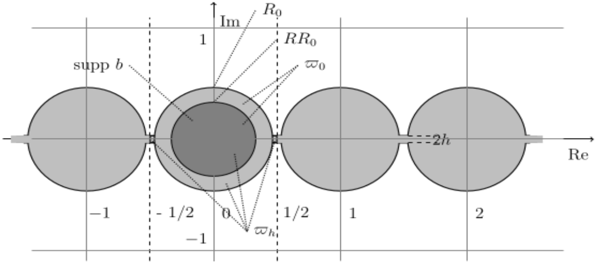
$$T_b : \sum_{n=0}^{\infty} f_n z^n \mapsto \sum_{n=0}^{\infty} b_n f_n z^n, \quad \text{where} \quad b_n = \pi^{-1}(n+1) \int_0^1 b(r) r^{2n+1} dr$$

Since the normalized monomials form an orthonormal basis of $A^2(\mathbb{D})$, we get that

$$\sigma(T_b) = \overline{\bigcup_{n \in \mathbb{N}} \{b_n\}} \subset \mathbb{R}$$

Goal: Construct a Toepliz operator T_a on a periodic domain, the **essential spectrum** of which is approximatively the same as $\sigma(T_b)$. In particular, $\sigma_{\text{ess}}(T_a)$ should contain many disjoint components. (cf. C.Sundberg, D.Zheng, Indiana University Mathematics Journal 59 (2010))

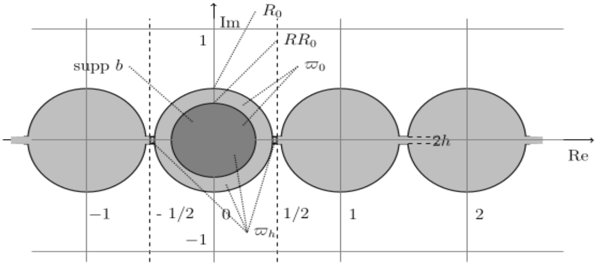
Toeplitz operator on $\mathbb{D}$ with interesting essential spectra



We consider a family of periodic domains Π_h, $h > 0$, with thin ligaments.

• With dilation $\tau : B(0, R_0) \to \mathbb{D}$, $R_0 \in (1/4, 1/2)$, we redefine b as $b \circ \tau$ and extend b to ϖ_h as zero. We define a periodic symbol a (uniquely) on Π such that it coincides with b on ϖ_h.

Toeplitz operator on $\mathbb{D}$ with interesting essential spectra



- For small $h > 0$, ϖ_h is a small domain perturbation of $B(0, R_0) \Rightarrow$

$$\sigma(T_{a,\eta}) \approx \sigma(T_b) \ \forall \eta \qquad \Rightarrow$$

$$\sigma_{\text{ess}}(T_a) = \bigcup_{\eta \in [-\pi, \pi]} \sigma(T_{a,\eta}) \approx \sigma(T_b)$$

The result on periodic domains

More precisely, the previous idea leads to the following result.

Theorem

Let $b \in L^\infty(\mathbb{D})$ be real valued and its support contained in $\mathbb{D}_R$ for some $R < 1$. Let the eigenvalues λ_n, $n \in \mathbb{N}$, of $T_b : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ be indexed such that

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots .$$

If $N \in \mathbb{N}$ and $\delta > 0$ are arbitrary numbers such that $|\lambda_N| > |\lambda_{N+1}| + 2\delta$, then for every small enough $\varepsilon > 0$ there exists a simply connected periodic domain $\Pi \subset \mathbb{C}$ and a bounded Toeplitz operator $T_a : A^2(\Pi) \to A^2(\Pi)$ such that

$$\sigma_{\rm ess}(T_a) \cap B(\lambda_n, \varepsilon) \neq \emptyset \ \forall \, n \leq N,$$

and

$$\mathrm{dist}\,\big(\sigma_{\rm ess}(T_a) \cap G_\varepsilon, \sigma_{\rm ess}(T_a) \setminus G_\varepsilon\big) \geq \delta, \quad \text{where } G_\varepsilon = \bigcup_{n \leq N} B(\lambda_n, \varepsilon).$$

Toeplitz operator on $\mathbb{D}$ with interesting essential spectra

By applying the Riemann mapping $\psi : \mathbb{D} \to \Pi$, the previous result implies the following version concerning Toeplitz operators on the disc.

Theorem

Given $K \in \mathbb{N}$ and any finite sequence of distinct real numbers $x_1 > \ldots > x_K$ one can find $\delta > 0$ such that for all small enough $\varepsilon > 0$, there exists a bounded Toeplitz-operator $T_a : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ with a real valued symbol $a \in L^\infty(\mathbb{D})$ and the properties

$$\sigma_{\mathrm{ess}}(T_a) \cap B(x_n, \varepsilon) \neq \emptyset \ \forall \, n \leq K$$

and

$$\mathrm{dist}\,\big(\sigma_{\mathrm{ess}}(T_a) \cap U_\varepsilon, \sigma_{\mathrm{ess}}(T_a) \setminus U_\varepsilon\big) \geq \delta, \quad \text{where } U_\varepsilon = \bigcup_{n \leq K} B(x_n, \varepsilon).$$

• In particular, given K, one can construct such an operator T_a with at least K disjoint components of $\sigma_{\mathrm{ess}}(T_a)$.

• We expect the components to be continua in stead of discrete sets, but this remains unproven. (The corresponding question in elliptic PDE's is classical and quite deep.)

Lemma on almost eigenvalues and -vectors

In the proofs of the previous theorems one uses,among other things, the following well-known lemma on almost eigenvalues and -vectors.

Lemma

Let $K : H \to H$ be a compact self-adjoint operator in a Hilbert-space H and let $\mu \in \mathbb{R}$. If there are $f \in H$ with $\|f\|_H = 1$ and $\delta > 0$ such that

$$\|Kf - \mu f\| \leq \delta,$$

then K has an eigenvalue $\lambda \in [\mu - \delta, \mu + \delta]$.

Bergman kernel in the simply connected case

We return to the considerations on the Bergman kernel, assume that ϖ is simply connected and show the connection of the kernel K_η with a certain Riemann mapping.

The exponential map $E : z \mapsto e^{i2\pi z}$ maps the set $\varpi \cup J_+ \cup J_-$ onto the doubly connected domain $\mathcal{D}$, which is contained in an annulus,

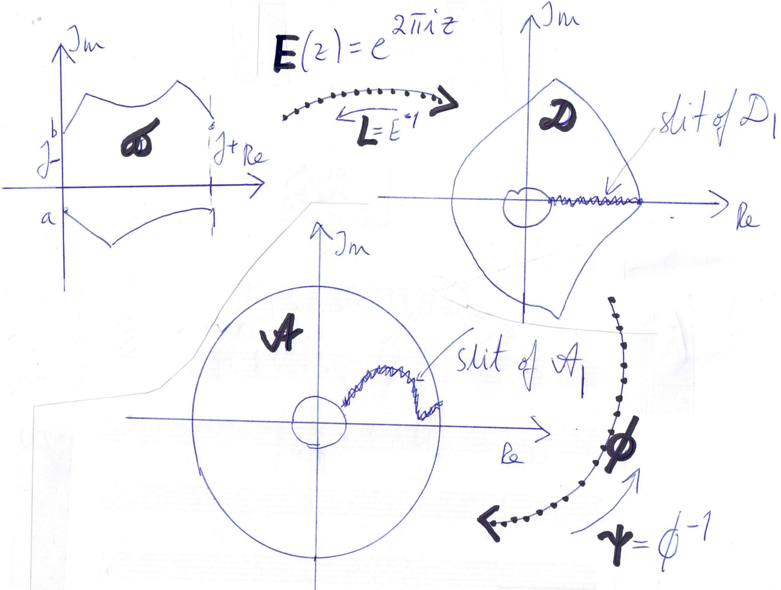
$$\mathcal{D} \subset \{z \in \mathbb{C} \,:\, \rho_0 < |z| < \rho_1\}.$$

Also, there exista a conformal mapping

$$\phi : \mathcal{D} \to \mathcal{A} = \{z \in \mathbb{C} \,:\, 1/\rho < |z| < \rho\}$$

where the number $\rho > 1$ is uniquely determined by $\mathcal{D}$. Denote $\psi = \phi^{-1}$.

Domains and mappings



We denote $\mathcal{D}_| = E(\varpi)$ and $\mathcal{A}_| = \phi(\mathcal{D}_|)$ and define on $\mathcal{A}_|$ the weight

$$V(z) = \frac{|\psi'(z)|^2}{4\pi^2|\psi(z)|^2} = v(z)\overline{v(z)} \quad \text{with} \quad v(z) = \frac{1}{2\pi}\frac{\psi'(z)}{\psi(z)};$$

note that v is analytic on $\mathcal{A}_|$. We denote by $A^2_{V,\eta}(\mathcal{A}_|)$ the closed subspace spanned by functions

$$z^{\eta/(2\pi)}g(z)\ , \quad g \in A^2_V(\mathcal{A})$$

Lemma

(i) The composition operator $I_1 : f \mapsto f \circ L \circ \psi$ is a unitary isomorphism $L^2(\varpi) \to L^2_V(\mathcal{A})$ and $A^2_\eta(\varpi) \to A^2_{V,\eta}(\mathcal{A}_|)$. (Here $L = (2\pi i)^{-1}\log z$)

(ii) An orthonormal basis of $A^2_{V,\eta}(\mathcal{A}_|)$ is formed by functions.

$$f_{n,\eta}(z) = C_{n,\eta} z^{n+\eta/(2\pi)} v(z)^{-1}, \quad n \in \mathbb{Z},$$

where $C_{n,\eta}$ are the normalization constants,

$$C^{-2}_{n,\eta} = \frac{2\pi}{2(n+1)+\eta/\pi}\left(\rho^{2(n+1)+\eta/\pi} - \rho^{-2(n+1)-\eta/\pi}\right).$$

Kernel K_η

The kernel $K_{\eta,\mathcal{A}}$ of the orthogonal projection $L^2_V(\mathcal{A}) \to \mathcal{A}^2_{V,\eta}(\mathcal{A}_|)$, i.e., $f \mapsto \int_{\mathcal{A}} K_{\eta,\mathcal{A}}(\cdot,w)f(w)dA(w)$ is

$$K_{\eta,\mathcal{A}}(z,w) = \sum_{n\in\mathbb{Z}} f_{n,\eta}(z)\overline{f_{n,\eta}(w)}V(w).$$

We denote $\quad \widetilde{K}(z,w) = 4\pi^2 e^{i2\pi(z-\bar{w})}\phi'(e^{i2\pi z})\overline{\phi'(e^{i2\pi w})}$

and define the conformal mapping $\varphi : \Pi \to S$, where S is the strip

$$S = (-\infty,\infty) \times \big(-(2\pi)^{-1}\log\rho, (2\pi)^{-1}\log\rho\big) \quad \text{and}$$

$$\varphi(z) = \frac{1}{i2\pi}\log\big(\phi(e^{i2\pi z})\big) + [\operatorname{Re} z] \quad \text{with} \quad \phi(e^{i2\pi z}) = e^{i2\pi\varphi(z)}.$$

Theorem

The kernel K_η of the projection from $L^2(\varpi)$ onto $A^2_\eta(\varpi)$ equals

$$K_\eta(z,w) = \widetilde{K}(z,w)\sum_{n\in\mathbb{Z}} \frac{2n+\eta/\pi}{2\pi(\rho^{2n+\eta/\pi} - \rho^{-2n-\eta/\pi})} e^{i(2\pi(n-1)+\eta)(\varphi(z)-\overline{\varphi(w)})}.$$

Kernel K_Π

Theorem

The kernel K_η of the projection from $L^2(\varpi)$ onto $A^2_\eta(\varpi)$ equals

$$K_\eta(z,w) = \widetilde{K}(z,w) \sum_{n\in\mathbb{Z}} \frac{2n+\eta/\pi}{2\pi\left(\rho^{2n+\eta/\pi} - \rho^{-2n-\eta/\pi}\right)} e^{i(2\pi(n-1)+\eta)(\varphi(z)-\overline{\varphi(w)})}.$$

We combine the previous kernel formula with the earlier general one:

$$K_\Pi(z,w) = \frac{\widetilde{K}(z,w)}{\pi} e^{-i2\pi(\varphi(z)-\overline{\varphi(w)}))} \int_{-\infty}^{\infty} \frac{t}{\rho^{2t}-\rho^{-2t}} e^{i2\pi t(\varphi(z)-\overline{\varphi(w)})} dt.$$

There holds the integral formula (Fourier transform)

$$\int_{-\infty}^{\infty} \frac{t}{2} \operatorname{csch}(at) e^{-ist} dt = \int_{-\infty}^{\infty} \frac{t}{e^{at}-e^{-at}} e^{-ist} dt = \frac{1}{4a^2} \pi^2 \operatorname{sech}^2\left(\frac{\pi s}{2a}\right),$$

where $s \in \mathbb{R}$ and $a > 0$ is a parameter and csch denotes the hyperbolic cosecant.

Kernel K_Π

We obtain

Theorem

If the periodic domain Π is simply connected then its Bergman kernel equals

$$K_\Pi(z,w) = \widetilde{K}_\Pi(z,w)\frac{\pi^3}{4(\log\rho)^2}\operatorname{sech}^2\Big(\frac{\pi^2(\varphi(z)-\overline{\varphi(w)})}{2\log\rho}\Big),$$

where

$$\widetilde{K}_\Pi(z,w) = e^{i2\pi(z-\varphi(z)-\bar{w}+\overline{\varphi(w)})}\phi'(e^{i2\pi z})\overline{\phi'(e^{i2\pi w})}$$

For the strip $\Sigma = (-\infty,\infty)\times(-\pi,\pi)$ the Bergman kernel can be computed and it is known to be

$$K_\Sigma(z,w) = \frac{1}{16\pi}\operatorname{sech}^2\big((z-\bar{w})/4\big)$$

which coincides with the above formula in this special case.

Weighted L^p-estimate

Application: a boundedness result for the Bergman projection with respect to certain weighted L^p-norms. Let us consider continuous weights $W : \Pi \to \mathbb{R}^+$ which only depend on the real part of the variable $z \in \Pi$. We assume that there are constants $a, C > 0$ and $0 < b < 1$ such that for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$

$$\frac{1}{C} W(x) e^{-a|n|^b} \leq |W(x+n)| \leq C W(x) e^{a|n|^b}. \tag{1}$$

Theorem

Let $W : \Pi \to \mathbb{R}^+$ be a weight as above and $1 \leq p < \infty$. Then, the projection operator $P_\Pi : L^p_W(\Pi) \to L^p_W(\Pi)$ is bounded.

FINALE

Thank you for your attention!