# Tensor series in Banach spaces

Jochen Wengenroth

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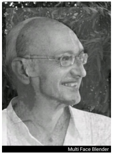

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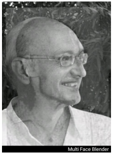


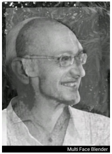

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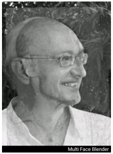

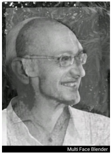

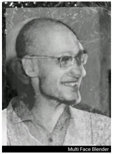

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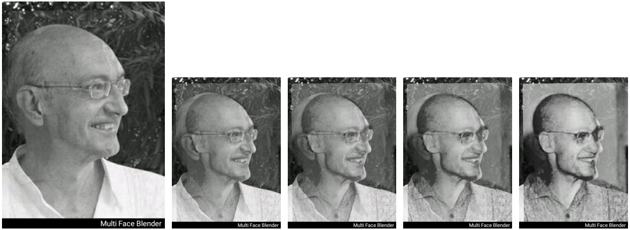

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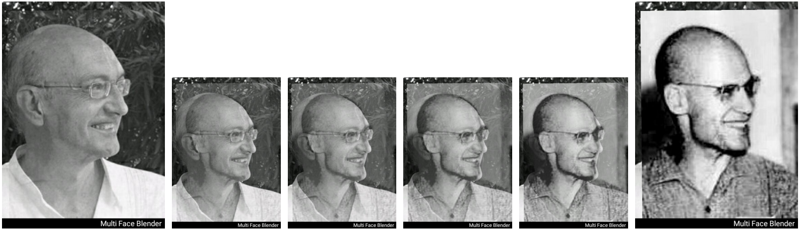

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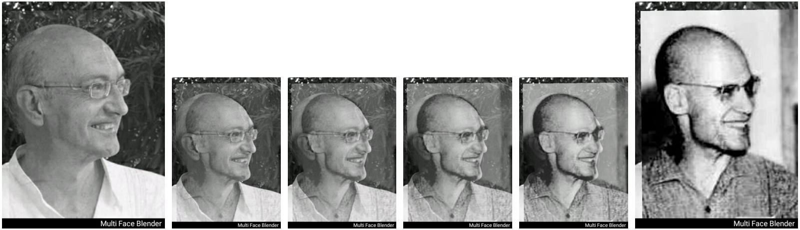


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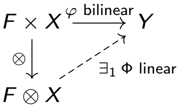


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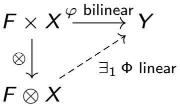


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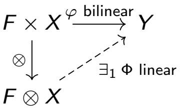


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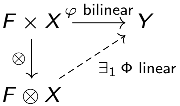


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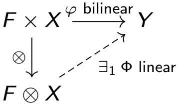


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However, $C(K) \hat{\otimes}_\pi X$ much smaller than $C(K, X)$.

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- IF the representation of $v_n$ above can be chosen such that $\alpha\left(\sum_{k=m(n-1)+1}^{p} f_k \otimes x_k\right) \leq 2\alpha(v_n)$ for all $m(n-1) < p \leq m(n)$ THEN $S_m$ converges to $u$ (write $m = m(n) + p$)

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- IF the representation of $v_n$ above can be chosen such that $\alpha\left(\sum_{k=m(n-1)+1}^{p} f_k \otimes x_k\right) \leq 2\alpha(v_n)$ for all $m(n-1) < p \leq m(n)$ THEN $S_m$ converges to $u$ (write $m = m(n) + p$)

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For every norm $\alpha$ on $F \otimes X$, every $v \in F \otimes X$ has a representation $v = \sum_{k=1}^{M} f_k \otimes x_k$ such that $\alpha\left(\sum_{k=1}^{p} f_k \otimes x_k\right) \leq 2\alpha(v)$ for all $1 \leq p \leq M$.

### Theorem

For every norm $\alpha$ on $F \otimes X$, every $u \in F\hat{\otimes}_\alpha X$ has a representation $u = \sum_{n=1}^{\infty} f_n \otimes x_n$.

- Successive approximation: $u = \sum_{n=1}^{\infty} v_n$ with $v_n \in F \otimes X$ such that $\sum_{n=1}^{\infty} \alpha(v_n) < \infty$.
- Represent $v_n = \sum_{k=m(n-1)+1}^{m(n)} f_k \otimes x_k$. Set $S_m = \sum_{k=1}^{m} f_k \otimes x_k$.
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Remark. For the projective norm this holds if $\sum_{k=1}^{M} \|f_k\|\|x_k\| \leq 2\pi(v)$. This proves Grothendieck's theorem even with absolute convergence of the tensor series.

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$$\alpha\left(\sum_{k=1}^{p} f_k \otimes x_k\right) \leq \frac{q}{N} \alpha(v) + \frac{1}{N} \sum_{j=1}^{r} \alpha(g_j \otimes y_j) \leq 2\alpha(v)$$

  if $N \geq n \max\{\alpha(g_k \otimes y_k) : 1 \leq k \leq n\} / \alpha(v)$.

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# Happy birthday, Pepe!